MATH 322-009 EXAM 1 Review Answers

1. $\left[\begin{array}{cc|c} h & 1 & 3 \\ 1 & 2 & K \end{array}\right]$ $\begin{array}{l} R_1 \to \frac{1}{h} R_1 \\ R_2 \to R_2 - \frac{1}{h} R_1 \end{array}$ $\left[\begin{array}{cc|c} 1 & \frac{1}{h} & \frac{3}{h} \\ 0 & 2-\frac{1}{h} & K-\frac{3}{h} \end{array}\right]$

a) $2 - \frac{1}{h} = 0$ and $K - \frac{3}{h} = 0$ reduces to $h = \frac{1}{2}$ $K = 6$ produces ∞ many solutions

b) $h = \frac{1}{2}$ $K \neq 6$ would produce no soln

c) $h \neq \frac{1}{2}$ will produce a unique solution

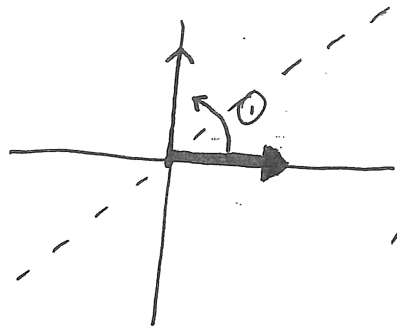
2. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ -x_3 + x_4 \\ x_2 + x_3 - 2x_4 \end{bmatrix}$ $e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $T(e_4) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ b/c $A = (T(e_1), T(e_2), T(e_3), T(e_4))$

3. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \vec{e}_1$

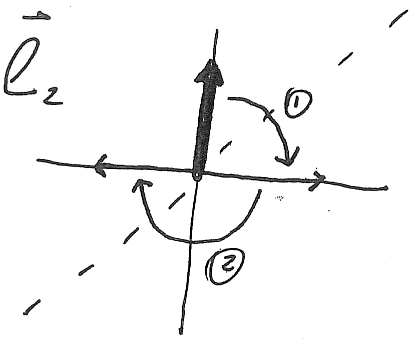


$\vec{e}_1 \qquad T\vec{e}_1$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Moves to y-axis after first reflection; second reflection does not affect it.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vec{e}_2$



$\vec{e}_2 \qquad T\vec{e}_2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Resulting matrix A: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

4. Since $Ax = b$ is consistent for every b in $\mathbb{R}^5$ then A is onto $\mathbb{R}^5$, $n \doteq m$. For A to be onto there must be a pivot in each row, since $m = n$, then the columns of A span $\mathbb{R}^5$. Since $m = 5$ the columns of A are linearly independent and A is one to one. Since A is one to one there can only be one solution for each b.

<u>Using Invertible Matrix Theorem:</u>

Since A is a square matrix, $m = n$, and has at least one solution to $Ax = b$ for every b in $\mathbb{R}^5$; then A is invertible. Since A is invertible it must have linearly independent columns and will be one to one, therefore there can only be one solution for each b.

5a. $\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

This goes from $\mathbb{R}^5 \to \mathbb{R}^3$

$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ is in $\mathbb{R}^5$, so it is not in image of T

5b. Is the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the image of T?

$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right]$

$x_1 - 2x_2 + 3x_5 = 1$

x_2 is free

$x_3 = 2$

$x_4 + 2x_5 = 3$

x_5 is free

$x_1 = 1 + 2x_2 - 3x_5$

x_2 is free

$x_3 = 2$

$x_4 = 3 - 2x_5$

x_5 is free

$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \vec{x}_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \vec{x}_5 \begin{bmatrix} -3 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

5) c) Is T one-to-one?

No, not every column has a pivot. The system has a free variable and that particular column is linearly dependent. To be one-to-one every column must be linearly independent.

5d. Yes, T is onto; A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if the columns of it span all of $\mathbb{R}^m$. In this case the columns of A do span all of $\mathbb{R}^3$, because there is a pivot in every row.

$\mathbb{R}^n = \mathbb{R}^5 \quad \mathbb{R}^m = \mathbb{R}^3$

6 Can a square matrix with 2 identical columns be invertible?

No, According to IMT, A^T is invertible. By taking the transpose you flip the rows and columns, now you have 2 identical rows. Also according to IMT, you need n pivots in an $n \times n$ matrix to be invertible. Impossible to get n pivots with 2 identical rows.

Ex.
$$M = \begin{bmatrix} A & A & C \\ A & A & C \\ A & A & C \end{bmatrix} \quad M^T = \begin{bmatrix} A & A & A \\ A & A & A \\ C & C & C \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$M^T = \begin{bmatrix} A & A & A \\ 0 & 0 & 0 \\ C & C & C \end{bmatrix} \to \text{impossible to get 3 pivots.}$$

7) Can a square matrix with two identical rows be invertible? Why or why not?

A Square matrix with two identical rows can't be invertible, given by the invertible matrix theorem (c). If two rows are identical then during reduction, one of those rows could be replaced by all zeros. This would result in not having n pivot positions. This violates (c) of the invertible matrix theorem: A has n pivot positions.