Math 322

Exam 2

Review Worksheet

These problems will help you review for exam 2. This is not a comprehensive review; it is just meant to help you get started.

$$\boxed{\text{SOLUTIONS}}$$

1. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 0 \end{bmatrix}$.

a. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}$. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$.

$$\hat{y} = \text{proj}_{\vec{v}} \vec{y} = \frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{2+0+3+0}{1+36+9+0}\begin{bmatrix} 1 \\ 6 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/46 \\ 30/46 \\ 15/46 \\ 0 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 5/46 \\ 15/23 \\ 15/46 \\ 0 \end{bmatrix} = \begin{bmatrix} 87/46 \\ -15/23 \\ 31/46 \\ 5 \end{bmatrix}$$

b. Find all vectors in $\mathbb{R}^4$ which are orthogonal to $\mathbf{v}$.

$$\begin{bmatrix} 1 \\ 6 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow x_1 + 6x_2 + 3x_3 + 0x_4 = 0$$

$$\Rightarrow x_1 = -6x_2 - 3x_3$$

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_2 - 3x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boxed{\text{Span}\left\{ \overset{x_2}{\begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{x_3}{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{x_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}}$$

2. In each case, decide if the matrix M is diagonalizable, is not diagonalizable, or if there is not enough information to decide.

(Note: $\lambda = 0$ means M is not invertible, but doesn't prevent diagonalizable.)

a. M is a 3×3 matrix with eigenvalues 0, 4, and 8.

IS diagonalizable: three distinct eigenvalues will yield three linearly ind. eigenvectors.

b. M is a 4×4 matrix with eigenvalues 1, 4 and 8.

Not enough information: depends on whether any eigenvalue has 2-dimensional eigenspace

c. M is a 5×5 matrix with three eigenvalues, two of which have 1-dimensional eigenspaces and one with a 2-dimensional eigenspace.

Not diagonalizable: dimensions of eigenspaces are $1 + 1 + 2 = 4$, which is less than 5, the # of columns of M.

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3. The matrix $A=\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1=2$, $\lambda_2=1$ and $\lambda_3=4$. Diagonalize A.

Hint: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$ are both eigenvectors.

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0+0 \\ 0+0+0 \\ 0+0+0 \end{bmatrix} = \underset{\lambda=2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 14+8+6 \\ 0-12-4 \\ 0+4+4 \end{bmatrix} = \begin{bmatrix} 28 \\ -16 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}. \quad — \lambda=4$$

We need an eigenvector for $\lambda=1$:

$$A-1I = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

x_3 is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\boxed{\text{Then } A=PDP^{-1} \text{ where } D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } P=\begin{bmatrix} 1 & -1 & 7 \\ 0 & 1 & -4 \\ 0 & 1 & 2 \end{bmatrix}}$$

Check work by verifying $AP=PD$.

↑ can choose any order for eigenvalues in diagonal

↑ as long as you choose the same order for corresponding columns of eigenvectors.

4. Let $A=\begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 3 & -1 \\ -2 & -4 & -6 & 2 \end{bmatrix}$. (Use your own paper for this problem.)

a. Find a basis for Col A.

b. Find a basis for Nul A.

Let $\mathbf{x}=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Verify $\mathbf{x}$ is in Nul A, and then find the coordinate vector for $\mathbf{x}$ in terms of your basis for Nul A.

c. Is your basis for Nul A orthogonal? Is it orthonormal? Explain briefly.

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④ $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 3 & -1 \\ -2 & -4 & -6 & 2 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Basis for Col A. In reduced version, 1st column has pivot, so basis is <u>original</u> 1st column, $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$.

(b) Basis for Nul A: use reduced version.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 + x_4 \\ y_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \text{Basis is } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\vec{x} = \begin{bmatrix} -4 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Verify $\vec{x}$ is in Nul A:

$$A\vec{x} = \begin{bmatrix} -4 + 2 + 3 - 1 \\ -4 + 2 + 3 - 1 \\ 8 - 4 - 6 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Coord. vector for $\vec{x}$:

$$\left[\begin{array}{ccc|c} -2 & -3 & 1 & -4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{Reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution:
$x_1 = 1$
$x_2 = 1$
$x_3 = 1$

Thus $\boxed{[x]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.}$ ← Basis has 3 vectors, so coord. vector has 3 components.

(c) Is the basis orthogonal?

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq 0.$$

No: not every pair (or, any pair) is perpendicular.

Is the basis orthonormal?

No: not orthogonal, so automatically No.
Also, length of vectors is not 1.

exam 2 review solutions

⑤ $W = \text{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where $\vec{x}_1 = \begin{bmatrix}1\\2\\-1\\0\end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix}2\\2\\0\\1\end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix}1\\1\\-1\\0\end{bmatrix}$

(a) Use Gramm-Schmidt to produce a new basis for W:

- the given vectors are linearly independent (trust me)

Step 1: $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix}1\\2\\-1\\0\end{bmatrix}$

Step 2: $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}\vec{x}_2 = \begin{bmatrix}2\\2\\0\\1\end{bmatrix} - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\begin{bmatrix}1\\2\\-1\\0\end{bmatrix}$

$= \begin{bmatrix}2\\2\\0\\1\end{bmatrix} - \frac{6}{6}\begin{bmatrix}1\\2\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ ← check: <u>is</u> orth. to $\vec{v}_1$

Step 3: $\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}\vec{x}_3 - \text{proj}_{\vec{v}_2}\vec{x}_3$

$$= \begin{bmatrix}1\\1\\-1\\0\end{bmatrix} - \frac{1+2+1}{1+4+1}\begin{bmatrix}1\\2\\-1\\0\end{bmatrix} - \frac{1+0-1+0}{1+1+1}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\-1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\2\\-1\\0\end{bmatrix} - 0 = \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\\0\end{bmatrix}$$

we can replace by any multiple, so use $\vec{v}_3 = \begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}$

New Basis: $\left\{ \begin{bmatrix}1\\2\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\0\end{bmatrix} \right\}$

(b) What property does the new basis have that the original does not?

it is orthogonal (dot prod. of any 2 is 0).