① $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec{v} \neq 0$. Find a basis for the set H of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to $\vec{v}$.

Solution: We know $\vec{v} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow ax + by + cz = 0$

for our class, this is enough:

$\Rightarrow ax = -by - cz$. If $a \neq 0$, we can write

$x = -\frac{b}{a} y - \frac{c}{a} z$, so $\begin{bmatrix} -\frac{b}{a} y - \frac{c}{a} z \\ y \\ z \end{bmatrix}$ ← with free variables y, z

gives a basis $\boxed{\left\{ \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix} \right\}}$ ← or any nonzero multiple of these.

to be thorough:

If $a = 0$ but $b \neq 0$, we have $by = -cz$

$\Rightarrow y = -\frac{c}{b} z$, so $\left\{ \begin{bmatrix} 0 \\ -\frac{c}{b} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ or any multiple of these

(x, z both free)

If $a = b = 0$ but $c \neq 0$, we have $cz = 0 \Rightarrow z = 0$

so a basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Since $\vec{v} \neq \vec{0}$, cannot have $a = b = c = 0$.

② Show that the mapping $T(\vec{x}) = \text{proj}_L \vec{x}$ is a linear transformation, where $L = \text{Span}\{u\}$ and $\vec{u} \neq \vec{0}$.

Solution: Must show two conditions

(a) $T(\vec{x} + \vec{y}) = \text{proj}_L(\vec{x} + \vec{y}) = \text{proj}_{\vec{u}}(\vec{x} + \vec{y}) = \dfrac{(\vec{x} + \vec{y}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

$= \dfrac{\vec{x} \cdot \vec{u} + \vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \left(\dfrac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} + \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$

$= \dfrac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} + \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \text{proj}_{\vec{u}} \vec{x} + \text{proj}_u \vec{y} = T(\vec{x}) + T(\vec{y})$

(b) Let c be a real number.

$$T(c\vec{x}) = \text{proj}_{\vec{u}}(c\vec{x}) = \frac{(c\vec{x})\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{c(\vec{x}\cdot\vec{u})}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= c\left(\frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\right) = c\cdot\text{proj}_{\vec{u}}\vec{x} = c\cdot T(\vec{x}).$$

③ Let $\vec{u}_1 = \begin{bmatrix}2\\1\\0\end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\vec{x} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

(a) Show $\vec{x}$ is not in Span$\{\vec{u}_1, \vec{u}_2\}$.

$$\left[\begin{array}{cc|c}2 & 1 & 0\\1 & -2 & 1\\0 & 3 & 0\end{array}\right] \xrightarrow[\frac{1}{3}R_3]{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c}1 & -2 & 1\\2 & 1 & 0\\0 & 1 & 0\end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{R_1 = R_1 + 2R_2} \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 0\\2 & 1 & 0\end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 2R_1} \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 0\\0 & 1 & -2\end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & -2\end{array}\right]$$

inconsistent system $\Rightarrow$ no lin. comb. of $\vec{u}_1, \vec{u}_2$ yields $\vec{x}$.

(b) Show $\vec{u}_1$ is orthogonal to $\vec{u}_2$ but $\vec{x}$ is not orthogonal to $\vec{u}_1$ or $\vec{u}_2$.

$\vec{u}_1 \cdot \vec{u}_2 = 2(1) + 1(-2) + 0(3) = 0$ orthogonal

$\vec{u}_1 \cdot \vec{x} = 2(0) + 1(1) + 0(0) = 1 \neq 0$ } not orthog.

$\vec{u}_2 \cdot \vec{x} = 1(0) + -2(1) + 3(0) = -2 \neq 0$

(c) Use $\vec{x}$ to construct $\vec{v}$ in $\mathbb{R}^3$ that is orthog. to $\vec{u}_1$ and $\vec{u}_2$.

$$\text{proj}_{\vec{u}_1}\vec{x} = \frac{\vec{u}_1\cdot\vec{x}}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 = \frac{1}{5}\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}2/5\\1/5\\0\end{bmatrix}$$

$$\text{proj}_{\vec{u}_2}\vec{x} = \frac{\vec{u}_2\cdot\vec{x}}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \frac{-2}{14}\begin{bmatrix}1\\-2\\3\end{bmatrix} = \begin{bmatrix}-1/7\\2/7\\-3/7\end{bmatrix}$$

add these:

$$\begin{bmatrix}\frac{2}{5} - \frac{1}{7}\\ \frac{1}{5} + \frac{2}{7}\\ 0 - \frac{3}{7}\end{bmatrix} = \begin{bmatrix}\frac{9}{35}\\ \frac{17}{35}\\ -\frac{3}{7}\end{bmatrix}$$

$$\vec{z} = \vec{x} - \left(\text{proj}_{\vec{u}_1}x + \text{proj}_{\vec{u}_2}x\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}9/35\\17/35\\-3/7\end{bmatrix} = \begin{bmatrix}-9/35\\18/35\\3/7\end{bmatrix} \text{ or } \begin{bmatrix}-9\\18\\15\end{bmatrix}$$

we want

mult. by 35

④ Use Gramm-Schmidt to find orthog. basis for Col A,

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

Solution: let $\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$

Can check these are linearly independent (they are), and that they are not already orthogonal (they're not).

Let $\vec{v}_1 = \vec{x}_1$. Then $\text{proj}_{\vec{v}_1} \vec{x}_2 = \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

$$= \frac{-6-24-2-4}{1+9+1+1} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -3 \\ -3 \end{bmatrix}$$

Let $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1} \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. ↙ $\vec{v}_2$

Now for $\vec{v}_3$:

$$\text{proj}_{\vec{v}_1} \vec{x}_3 = \frac{-6+9+6-3}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\text{proj}_{\vec{v}_2} \vec{x}_3 = \frac{18+3+6+3}{9+1+1+1} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 15/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$$

$\vec{v}_3$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1} \vec{x}_3 - \text{proj}_{\vec{v}_2} \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 15/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$\boxed{\text{New basis: } \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}.}$$

← Can verify all pairs are orthogonal.