Written HW 8

$\frac{20}{20}$

1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

a. Show that the columns of A form an orthogonal basis for Col A

$\frac{1}{1}$ $\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 0$ $\quad \vec{v}_1 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0$ $\quad \vec{v}_2 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0$

The dot product of the columns of A are zero, so they are orthogonal, and because they are orthogonal they are linearly independent.

b. Find the least-squares solution of $A\vec{x} = \vec{b}$ by finding $\hat{b}$, the orthogonal projection of $\vec{b}$ onto Col A, and then solving $A\vec{x} = \hat{b}$.

$\frac{4}{4}$

$$\text{proj}_{\text{Col}A}\vec{b} = \frac{\vec{b}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix}$$

$$+ \frac{\vec{b}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix}$$

$$+ \frac{\vec{b}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3 = \frac{-5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \hat{b}$$

$$A\vec{x} = \hat{b} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ -1 & 1 & -1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 14/3 \\ 0 & 0 & 1 & -5/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \vec{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$$

c. Find the least-squares solution by solving $A^TA\vec{x} = A^T\vec{b}$

$\frac{3}{3}$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 14 \\ 0 & 0 & 3 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 14/3 \\ 0 & 0 & 1 & -5/3 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$$

2. Suppose we collected data associating a variable x with a variable y, and found the data set $(1,7), (2,4), (4,3), (6,1)$

a. Set up the design matrix, parameter vector and target vector to find the least squares line of best fit, $y = b_0 + b_1 x$

2/2

$$7 = b_0 + 1b_1$$
$$4 = b_0 + 2b_1$$
$$3 = b_0 + 4b_1$$
$$1 = b_0 + 6b_1$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

b. Set up the design matrix, parameter vector and target vector to find the least squares line of best fit $y = b_0 + b_1 x + b_2 x^2$

2/2

$$7 = b_0 + 1b_1 + 1b_2$$
$$4 = b_0 + 2b_1 + 4b_2$$
$$3 = b_0 + 4b_1 + 16b_2$$
$$1 = b_0 + 6b_1 + 36b_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

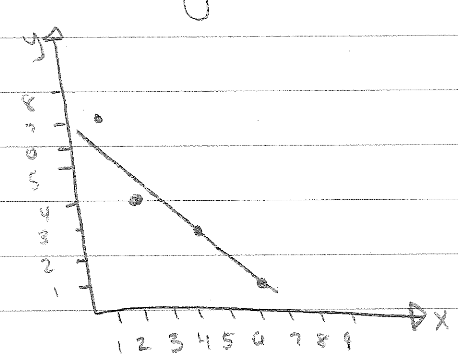
c. Solve for the lines of best fit for a and b then draw a graph that shows the data points and the line equation and another graph with the data points and the quadratic equation.

2/2

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ 13 & 57 \end{bmatrix} \qquad A^T\vec{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 33 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 13 & 15 \\ 13 & 57 & 33 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 426/59 \\ 0 & 1 & -63/59 \end{array}\right] \qquad \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 426/59 \\ -63/59 \end{bmatrix}$$

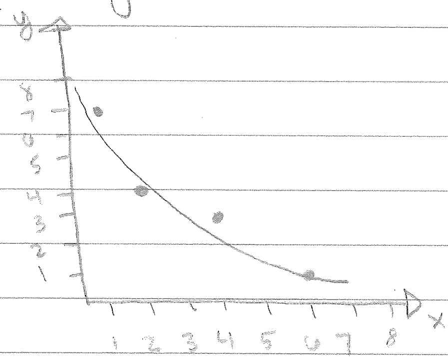
$$y = 426/59 - 63/59\, x$$



$$A^TA=\begin{bmatrix}1&1&1&1\\1&2&4&6\\1&4&16&36\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&4\\1&4&16\\1&6&36\end{bmatrix}=\begin{bmatrix}4&13&57\\13&57&289\\57&289&1569\end{bmatrix}\qquad A^T\vec{y}=\begin{bmatrix}1&1&1&1\\1&2&4&6\\1&4&16&36\end{bmatrix}\begin{bmatrix}7\\4\\3\\1\end{bmatrix}=\begin{bmatrix}15\\33\\107\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}4&13&57&15\\13&57&289&33\\57&289&1569&107\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&0&0&1693/199\\0&1&0&-1687/796\\0&0&1&119/796\end{array}\right]$$

$$y=\frac{1693}{199}-\frac{1687}{796}x+\frac{119}{796}x^2$$



The quadratic data looks better because the line sits closer to the data points.

3. Let $T: \mathbb{P}_2 \to M_{2\times2}$ be a linear transformation defined by $T(ax^2+bx+c)=\begin{bmatrix}a+b & a+c\\ a+c & a+b\end{bmatrix}$

a. Find $\ker T$ $\quad a+b=0 \quad a=-b$

3/3 $\quad a+c=0 \quad a=-c$

$\ker T=\{ax^2-ax-a \mid a \text{ is real}\}$

b. image of T

3/3 $\begin{bmatrix}a+b & a+c\\ a+c & a+b\end{bmatrix}=\begin{bmatrix}w & x\\ y & z\end{bmatrix}$

$a+b=w=z$

$a+c=x=y$

image of $T=\left\{\begin{bmatrix}w & y\\ y & w\end{bmatrix} \,\middle|\, w \text{ and } y \text{ are real}\right\}$

20/20

4-14-17
MA 322-009

Written Homework # 8

1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

a. Show that the columns of A form an orthogonal basis for Col A.

First we will show linear independance

$$\begin{array}{ccc} \vec{x_1} & \vec{x_2} & \vec{x_3} \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[R_4 = R_4 + R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow[R_2 = R_2 + R_3]{\text{Switch } R_2 \,\&\, R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we have a pivot in every column, all the columns are linearly independent

1/1

$\vec{x_1} \cdot \vec{x_2} = (1)(1) + (1)(0) + (0)(1) + (-1)(1) = 1 - 1 = 0$
$\vec{x_1} \cdot \vec{x_3} = (1)(0) + (1)(-1) + (0)(1) + (-1)(-1) = -1 + 1 = 0$
$\vec{x_2} \cdot \vec{x_3} = (1)(0) + (0)(-1) + (1)(1) + (1)(-1) = 1 - 1 = 0$

Therefore the columns of A form an orthogonal basis for Col A since the columns are linearly independent and the dot product of each is 0.

b. Find the least squares solution of $A\vec{x} = \vec{b}$ by finding $\hat{b}$, the orthogonal projection of b onto Col A, and then solving $A\vec{x} = \hat{b}$

$$\text{proj}_{\vec{x_1}} \vec{b} = \frac{\vec{b} \cdot \vec{x_1}}{\vec{x_1} \cdot \vec{x_1}} \vec{x_1}$$

$$= \frac{(2)(1) + (5)(1) + (6)(0) + (6)(-1)}{1^2 + 1^2 + 0^2 + (-1)^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix}$$

$$\text{proj}_{\vec{x_2}} \vec{b} = \frac{(2)(1) + (5)(0) + (6)(1) + (6)(1)}{1^2 + 0^2 + 1^2 + 1^2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= 14/3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix}$$

$$\text{proj}_{\vec{x_3}} \vec{b} = \frac{(2)(0) + (5)(-1) + (6)(1) + (6)(-1)}{(-1)^2 + 1^2 + (-1)^2} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= -5/3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix}$$

$$\hat{b} = \text{proj}_{\vec{x_1}} \vec{b} + \text{proj}_{\vec{x_2}} \vec{b} + \text{proj}_{\vec{x_3}} \vec{b}$$

$$\hat{b} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

Now solve $A\vec{x} = \hat{b}$

→ Since we have an orthogonal basis, the weights of each projection should make up our $\vec{x}$

→ Therefore $\vec{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$

4/4

c. Now find the least-squares solution another way by solving $A^T A \vec{x} = A^T \vec{b}$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

$$\begin{array}{ccc} \vec{x_1} & \vec{x_2} & \vec{x_3} \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 14 \\ 0 & 0 & 3 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 14/3 \\ 0 & 0 & 1 & -5/3 \end{array}\right]$$

3/3

So $\vec{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$ The same answer as in part b. except this way was much easier to compute!

2. Suppose we collected data associating a variable x with a variable y, and found the data set: $(1,7), (2,4), (4,3), (6,1)$

a. Set up the design matrix, parameter vector, and target vector to find the least squares line of best fit, $y = b_0 + b_1 x$

2/2

Design Matrix A: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix}$

Parameter Vector β: $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$

Target vector $\vec{y}$: $\begin{bmatrix} 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$

$A\vec{b} = \vec{y}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ 13 & 57 \end{bmatrix}$$

$$A^T \vec{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 33 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 13 & 15 \\ 13 & 57 & 33 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 4 & 13 & 15 \\ 0 & 59/4 & -63/4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & 13 & 15 \\ 0 & 1 & -63/59 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 4 & 0 & 1704/59 \\ 0 & 1 & -63/59 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7.220 \\ 0 & 1 & -1.068 \end{array}\right]$$

Therefore: The least-squares line of best fit is: $y = 7.220 - 1.068x$

b. Set up the design matrix, parameter, vector, and target vector to find the least-squares quadratic of best fit, $y = b_0 + b_1 x + b_2 x^2$

2/2

Design Matrix: A: $\begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 4 & 4^2 \\ 1 & 6 & 6^2 \end{bmatrix}$ Parameter vector: β: $\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$

Target vector: $\vec{y}$ $\begin{bmatrix} 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ $A\vec{b} = \vec{y}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \\ 1 & 4 & 16 & 36 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix} = \begin{bmatrix} 4 & 13 & 57 \\ 13 & 57 & 289 \\ 57 & 289 & 1569 \end{bmatrix}$$

$$A^T \vec{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \\ 1 & 4 & 16 & 36 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 33 \\ 107 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 13 & 57 & 15 \\ 13 & 57 & 289 & 33 \\ 57 & 289 & 1569 & 107 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1693/199 \\ 0 & 1 & 0 & -1687/796 \\ 0 & 0 & 1 & 119/796 \end{array}\right]$$

$$\beta = \begin{bmatrix} \frac{1693}{199} = 8.508 \\ \frac{-1687}{796} = -2.119 \\ \frac{119}{796} = 0.149 \end{bmatrix}$$

Quadratic line of best fit is:

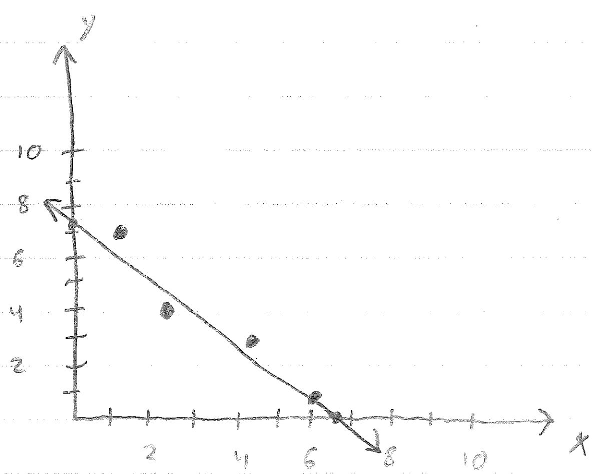
$$y = 8.508 - 2.119x + 0.149x^2$$

c. Use technology to solve for the line and quadratic you set up in parts (a) and (b). Then draw a graph that shows the data points and the line equation you found, and another graph w/ the data points and the quadratic you found. Which looks like it fits the data better?

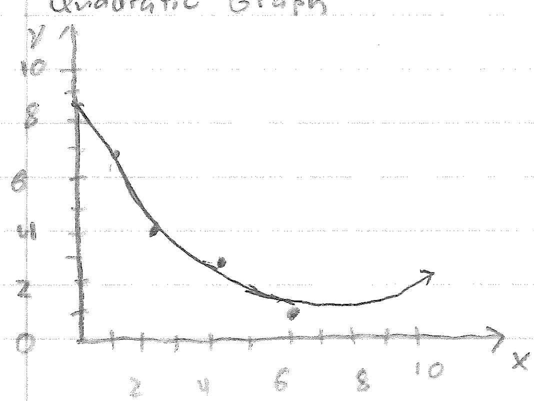
2/2

→ Solutions are in parts (a) and (b) of my response for each

Line graph



Quadratic Graph



• The line graph appears to fit the data better, but the quadratic is actually a better representation as it's {y-value} value at each corresponding x-value is actually closer to the data we have been given than the line. However, with only 4 data points it's difficult to determine if a quadratic or a line would fit the data better if more data points had been sampled.

3. Let $T: \mathbb{P}_2 \to M_{2\times 2}$ be a linear transformation defined by $T(ax^2+bx+c) = \begin{bmatrix} a+b & a+c \\ a+c & a+b \end{bmatrix}$

a. Find ker T

→ The kernel, or null space, of T is the set of all $\vec{u}$ in V such that $T(\vec{u}) = \vec{0}$.

$\cancel{T(\vec{u}) = 0 \quad \left[\begin{array}{cc|c} a+b & a+c & 0 \\ a+c & a+b & 0 \end{array}\right]}$

~~Row 1:~~ $\cancel{a+b = -a-c = -(a+c)}$

~~Row 2:~~ $\cancel{a+c = -a-b = -(a+b)}$

~~So~~ $\cancel{a+b = -(a+c)}$

$\cancel{2a = -b-c}$

$\cancel{a = -\tfrac{1}{2}b - \tfrac{1}{2}c}$

~~2 free variables, result in~~ $\cancel{\mathbb{P}_3}$

~~ker T =~~ $\cancel{\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}}$

~~b. Find the image of T {also called the range}~~

~~→ essentially is the column space of the matrix~~

3/3

We want: $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad a+b=0 \quad a+c=0$
$\quad a+c=0 \quad a+b=0$

$a=-b \qquad a=-c$
$b=-a \qquad c=-a$

Now find the kernel, we want a polynomial

$\ker T = \left\{ ax^2 + bx + c \;\middle|\; \begin{matrix} b = -a \\ c = -a \end{matrix},\ a \text{ is real} \right\}$

$\ker T = \{ ax^2 - ax - a \mid a \text{ is real} \}$

b. Find the image of T {also called the range}.

3/3

$\begin{bmatrix} m & n \\ o & p \end{bmatrix}$ Of all possible 2x2 matrices, which can we have?

$m = a + b = p$ $\qquad$ $m = p$

$o = a + c = n$ $\qquad$ $o = n$

$m = a + b$ $\qquad$ $n = a + c$ $\qquad$ $n = m - b + n - a$

$a = m - b$ $\qquad$ $n = m - b + c$ $\qquad$ $n = m - (m - a) + n - a$

$n = n \rightarrow$ circular

Range : $\left\{ \begin{bmatrix} m & n \\ n & m \end{bmatrix} \;\middle|\; m, n \text{ are real} \right\}$