MA 322

$\frac{24}{24}$

HW 5

1) Usually $\det(A+B) \neq \det(A) + \det(B)$. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that $\det(A+B) = \det(A) + \det(B)$ iff $a+d=0$

$\frac{5}{5}$

↳ This proof can be shown directly. We 1st evaluate $A+B$:

$$A+B = \begin{bmatrix} 1+a & b \\ c & 1+d \end{bmatrix}, \quad \text{so } \det(A+B) = (1+a)(1+d) - bc$$
$$= 1+a+d+ad-bc$$

↳ Then we evaluate $\det(A)+\det(B) = [1\cdot 1 - 0\cdot 0] + [ad-bc]$
$$= 1+ad-bc$$

• These 2 determinants are supposed to equal each other:
$1+a+d+ad-bc = 1+ad-bc$, & they only equal each other when $a+d=0$, so the relation now becomes:

$$1+0+ad-bc = 1+ad-bc$$
$$1+ad-bc = 1+ad-bc$$, these are clearly equivalent & so our assumption that $\det(A+B) = \det(A)+\det(B)$ iff $a+d=0$ is proved true

2.) A & P are $n \times n$ & P is invertible. Show $\det(PAP^{-1}) = \det(A)$

$\frac{4}{4}$

↳ Given $\det(AB) = \det(A)\cdot\det(B)$, & $\det(A^{-1}) = 1/\det(A)$ we can prove the given statement

↳ $\det(PAP^{-1}) = \det(PA)\cdot\det(P^{-1})$ as $\det(AB) = \det(A)\cdot\det(B)$.
And $\det(P^{-1}) = 1/\det(P)$ so $\det(PA)\cdot\det(P^{-1}) = \det(PA)\cdot 1/\det(P)$

↳ $\det(PA)\cdot 1/\det(P) = \det(P)\cdot\det(A)\cdot\frac{1}{\det(P)}$. And

$\det(P)\cdot 1/\det(P) = 1$ so we have:

$$\overbrace{\det(P)\cdot 1/\det(P)}^{=1}\cdot\det(A) = \det(A)$$
$\det(A) = \det(A)$, which is what we were trying to prove

3) A is $n \times n$ w/ $A^T \cdot A = I_n$. What are the possible values for $\det(A)$?

5/5

$\hookrightarrow$ We can take the determinants of both sides & get $\det(A^T \cdot A) = \det(I_n)$. And the left side can be rewritten as $\det(A^T) \cdot \det(A)$. And $\det(A^T) = \det(A)$ so we have $\det(A) \cdot \det(A) = \det(I_n)$. I_n is a triangular matrix so $\det(I_n)$ is the product of the main diagonal, so $\det(I_n) = 1^n = 1$.

$\hookrightarrow$ We now have $\det(A) \cdot \det(A) = 1$. Clearly $\det(A)$ can either be 1 or -1 as these are the only 2 values which have squares equal to 1.

4.) Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 4$ for the matrix:

$$A = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}$$

• The eigenspace is just the basis of the Null space of $(A - \lambda I_n)\vec{x} = \vec{0}$, which will be $(A - 4I_n)\vec{x} = \vec{0}$

$\longrightarrow$ $(A - 4I_n)$ is found by subtracting 4 from the diagonal of A:

$$(A - 4I_n) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 4 & -2 & -2 & 0 \end{bmatrix}, \text{ \& then we just solve } (A - 4I_n)x = 0$$

$\longrightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 4 & -2 & -2 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \cdot = -1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 4 & -2 & -2 & 0 & 0 \end{array}\right] \xrightarrow{R_2 += R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 4 & -2 & -2 & 0 & 0 \end{array}\right] \xrightarrow{R_3 += -R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 4 & -2 & -2 & 0 & 0 \end{array}\right] \xrightarrow{R_3 += -\frac{1}{2} R_3}$$

6/6

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 4 & -2 & -2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 & x_2 are basic variables & x_3 & x_4 are free

$x_1 = x_3$

$x_2 = x_3$

$x_3 =$ free

$x_4 =$ free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

$\hookrightarrow$ The basis for this eigenspace is then the following: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

5) Construct a 2×2 w/ 1 distinct eigenvalue.

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, We find the eigenvalue by finding $\det(A - \lambda I_n) = 0$

$$\det(A - \lambda I_n) = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - 0 = 0$$

$\longrightarrow$

• $(2-\lambda)(2-\lambda)=0$

$\lambda=2$, so the matrix $A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ has only the eigenvalue 2

6.) A formula exists to find det(A), w/ A a 3x3 matrix.

a.) Read 3.1 & do Problems 15 & 17 (not to be turned in).

b.) Use this technique to find the characteristic eqn for

$$A=\begin{bmatrix}0&3&1\\3&0&2\\1&2&0\end{bmatrix}, \qquad A-\lambda I_n=\begin{bmatrix}-\lambda&3&1\\3&-\lambda&2\\1&2&-\lambda\end{bmatrix}$$

$$\det(A-\lambda I_n)=-\lambda\left[(-\lambda\cdot-\lambda)-(2\cdot 2)\right]-3\left[(3\cdot-\lambda)-2\right]+\left[6+\lambda\right]$$

$$=-\lambda(\lambda^2-4)-3(-3\lambda-2)+6+\lambda$$

$$=-\lambda^3+4\lambda+9\lambda+6+6+\lambda$$

$$=\boxed{-\lambda^3+14\lambda+12=0}$$