1. (10 points) Give the parametric equations for the line in the xy-plane which passes through the points $(-2, 9)$ and $(4, 12)$.

$y - 12 = \frac{1}{2}(x - 4)$

$y = \frac{1}{2}x + 10$

So $x(t) = t$

$y(t) = \frac{1}{2}t + 10$

2. The curve W is given by the parametric equations:

$$x(t) = e^{4t}, \ y(t) = \sin(2\pi t).$$

(a) (10 points) Find $\frac{dy}{dx}$ and express it as a function of t.

$\frac{dx}{dt} = 4e^{4t} \neq 0$

$\frac{dy}{dt} = 2\pi \cos(2\pi t)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\pi \cos(2\pi t)}{4 e^{4t}}$

$= \frac{\pi e^{-4t} \cos(2\pi t)}{2}$

(b) (10 points) Give the parametric equations for the tangent line to W at the point where $t = 1/4$.

$x(1/4) = e$ $\quad y(1/4) = 1$

$\frac{dx}{dt}(1/4) = 4e^1 = 4e$

$\frac{dy}{dt}(1/4) = 2\pi \cdot 0 = 0$

Tangent line

$y - 1 = \frac{0}{4e} \cdot (x - e)$

$y = 1$

So $x(t) = 0$

$y(t) = 1$

3. (10 points) Determine if the series

$$\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$$

is convergent or divergent.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(n+1)!/(n+3)!}{2^n(n)!/(n+2)!}\right| = \lim_{n\to\infty} \left|\frac{2(n+1)}{n+3}\right|$$

$$= 2\lim_{n\to\infty} \left|\frac{n+1}{n+3}\right| = 2 > 1$$

So $\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$ diverges by ratio test.

4. (10 points) Determine if the series

$$\sum_{k=1}^{\infty} \frac{1}{2+3^k}$$

is convergent or divergent.

$$\frac{1}{2+3^k} < \frac{1}{3^k} = \left(\frac{1}{3}\right)^k$$

Since $\sum \left(\frac{1}{3}\right)^k$ is convergent geometric series,

$\sum \frac{1}{2+3^k}$ converges by comparison test.

5. (10 points) Find the radius of convergence and interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n}.$$

$a_n = \frac{(x-6)^n}{n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-6)^{n+1}/n+1}{(x-6)^n/n}\right| = \lim_{n\to\infty} \left|\frac{(x-6)n}{n+1}\right|$

$= |x-6| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |x-6|$

So converges when $|x-6|<1$ ie. $5<x<7$.

So $\boxed{R=1.}$

when $x=5$

$\sum a_n = \sum \frac{(-1)^n}{n}$ converges by AST.

when $x=7$

$\sum a_n = \sum \frac{1}{n}$ diverges as harmonic

So $I=[5,7)$.

6. (2 points each) Answer each of the following questions either True or False. You must write the entire word to get full credit. You do not need to show any work on this problem.

   (a) To determine if the series $\sum_{n=1}^{\infty} \frac{1+\sin(n\pi/2)}{n}$ converges, you could apply the integral test.

   False

   (b) If $\sum |a_n|$ is divergent then $\sum a_n$ is divergent.

   False

   (c) The series $\sum_{n=1}^{\infty} (-1)^n/n$ is conditionally convergent.

   True

   (d) The curve represented by the parametric equations $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$ is a circle.

   True

   (e) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.

   False

7. (10 points each) Find a power series representation for the functions below. Do NOT determine their intervals of convergence.

(a) $f(x) = \frac{1}{1+9x^2}$.

$$f(x) = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty} (-9x^2)^n \qquad |-9x^2| < 1$$

$$= \sum_{n=0}^{\infty} (-9)^n x^{2n} \qquad |-9x^2| < 1$$

(b) $f(x) = \ln(5 - x)$.

$$f'(x) = \frac{-1}{5-x} = \frac{-1}{5} \cdot \frac{1}{1-\frac{x}{5}}$$

$$= \frac{-1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{-1}{5^n} x^n = -1 + \frac{x}{5} - \frac{x^2}{25} + \cdots$$

$$f(x) = \int f'(x)dx = C - x + \frac{x^2}{5 \cdot 2} - \frac{x^3}{5^2 \cdot 3} + \cdots$$

$$f(0) = \ln(5) = C. \quad \text{So } f(x) = \ln(5) + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{5^n(n+1)}$$

8. (a) (3 points) What is the Maclaurin series expansion of e^x? You do not have to justify your work.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(b) (7 points) Using part (a) of this problem, determine the Taylor series for the function:

$$g(x) = (x-3)^2 e^x$$

centered at $a = 3$.

$$g(x) = (x-3)^2 f(x) \qquad f^{(n)}(x) = e^x$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

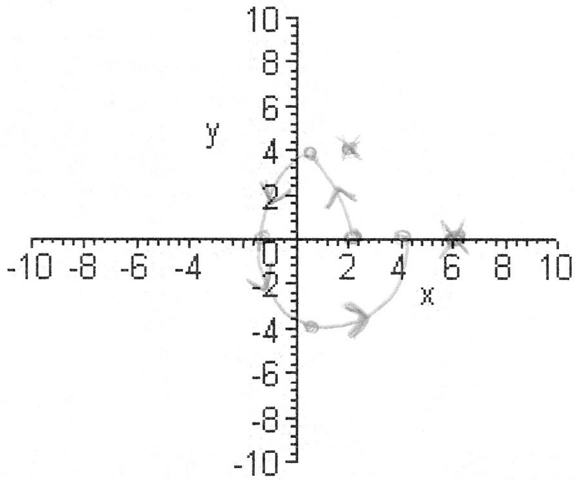
$$\text{So } g(x) = (x-3)^2 \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^{n+2}$$

9. Extra Credit: (5 points) Consider the parametric equations:

$$x = t/\pi + 2\cos t,\ y = 4\sin t,\ 0 \le t \le 2\pi.$$

On the axis below, graph the curve C described by the above parametric equations and indicate with arrows the direction that the curve is traced.



t	x	y
0	2	0
$\pi/6$		
$\pi/4$		
$\pi/3$		
$\pi/2$	$1/2$	4
π	-1	0
$3\pi/2$	$3/2$	-4
2π	4	0

The equations $x = 2\cos t$, $y = 4\sin t$ give an ellipse with $R = 4$, $r = 2$.

The t/π in $x(t)$ means r is increasing as $t \to 2\pi$

So we gain a spiral.