MA 322 - Matrix Algebra

Exam 2

12 March 2009

Name:____________________________

Score:_________________ /100 Points

Instructions:

- You may not use any outside assistance on this exam. You may not use books, notebooks, other people's exams, or any other materials to cheat on this exam.
- You may not use a graphing calculator on this exam.
- The use of electronic equipment such as mp3 players, ipods, cell phones and other electronic devices during the exam is prohibited.
- If you are caught cheating on the exam, you will be given a 0 for a grade.
- Write clearly during the exam and fully erase or mark out anything you do not want graded.
- You must give exact answers and fully reduce fractions to receive full credit. Approximate and unreduced answers will receive only partial credit.
- **You must show all your work to receive full credit unless otherwise stated.**

1. (5 points each) Consider the matrix C and the row echelon form of C below.

$$C = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Determine a basis for the column space of C.

Solution: A basis for the $ColC$ is given by the pivot columns of C. Thus a basis is given by the set:

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ -6 \end{bmatrix} \right\}.$$

(b) What is the rank of C?

Solution: The rank of C is the dimension of $ColC$ which is 3 by last solution.

(c) Determine a basis for the nullspace of C.

Solution: We know if we row reduce the augmented matrix $[C\ 0]$ we get:

$$[C\ 0] \sim \begin{bmatrix} 1 & 2 & 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus we have the system of equations:

$$\begin{aligned} x_1 &= -2x_2 + 5x_4 \\ x_3 &= 2x_4 \\ x_5 &= 0. \end{aligned}$$

Hence a solution to the equation $Cx = 0$ is of the form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

Hence a basis for the nullspace of C is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(d) What is the dimension of $nulC$? The dimension of $nulC = 2$.

2. (5 points each) Determine if the following are true or false and write the corresponding full word for your answer. If you write simply 'T' or 'F' you will receive NO credit. No justification is necessary on this problem.

   (a) An $n \times n$ matrix A is invertible if and only if the number 0 is an eigenvalue of A.

   False

   (b) If B and C are $n \times n$ matrices then $detBC = (detB)(detC)$.

   True

   (c) The dimensions of $ColD$ and $NulD$ add up to the number of columns of D.

   True

   (d) Suppose a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has the property that $T(u) = T(v)$ for some pair of distinct vectors u and v. Then T could map $\mathbb{R}^3$ onto $\mathbb{R}^3$.

   False

   (e) If v_1 and v_2 are vectors in $\mathbb{R}^n$, then $Span\{v_1, v_2\}$ is always a subspace of $\mathbb{R}^n$.

   True

3. (5 points each) Let $B = \begin{bmatrix} 1 & 0 \\ -6 & -1 \end{bmatrix}$.

(a) What is the characteristic equation for B?

Solution:

$$\begin{aligned} det(B - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 \\ -6 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda) \\ &= -1+\lambda^2 \end{aligned}$$

(b) Determine all of the eigenvalues for B.

Solution: The eigenvalues of a triangular matrix are the diagonal entries. So the eigenvalues are $\lambda = \pm 1$.

(c) Determine a basis for each eigenspace.

Solution: For $\lambda = 1$, we see: $[B - I] = \begin{bmatrix} 0 & 0 \\ -6 & -2 \end{bmatrix}$. Thus row reducing the augmented matrix $[B - I\ 0]$ to REF we see:

$$[B - I\ 0] \sim \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the eigenspace is 1-dimensional with a basis given by the vector: $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$.

By a similar process, the eigenspace for $\lambda = -1$ is 1-dimensional with a basis given by the vector: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(d) Is B diagonalizable? If yes, determine P, P^{-1}, and D such that $B = PDP^{-1}$.

Solution: Since there are 2 distinct eigenvectors and B is 2×2, we know that

$$P = \begin{bmatrix} -1/3 & 0 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Using the formula to find determinants for a 2×2-matrix we see:

$$P^{-1} = \begin{bmatrix} -3 & 0 \\ 3 & 1 \end{bmatrix}.$$

(e) Using Part 3d, determine B^3.

Solution: $B^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}$. Hence:

$$\begin{aligned} B^3 &= \begin{bmatrix} -1/3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1/3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -6 & -1 \end{bmatrix} \end{aligned}$$

4. (10 points) Using cofactor expansion and row reduction, determine the determinant of the matrix:

$$A = \begin{bmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ -1 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{bmatrix}.$$

Solution:

$$\begin{aligned} detA &= \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 7 & 8 & 14 & 0 \\ 4 & 2 & 4 & 3 \end{vmatrix} \\ &= 3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ -9 & 0 & -2 \end{vmatrix} \\ &= 3\begin{vmatrix} -1 & 2 & 3 \\ 5 & 0 & -3 \\ -9 & 0 & -2 \end{vmatrix} \\ &= (3)(-2)\begin{vmatrix} 5 & -3 \\ -9 & -2 \end{vmatrix} \\ &= (3)(-2)(-10-27) \\ &= (-6)(-37) = 222 \end{aligned}$$

5. (10 points each) Consider the set of vectors $S = \left\{ \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -14 \end{bmatrix} \right\}$.

(a) Determine a basis $\mathcal{B}$ for S.

Solution: Since we see the first two vectors are linearly independent as one is not a multiple of the other but:

$$2 \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ -14 \end{bmatrix}.$$

Thus a basis of S is given by the first two vectors, call them b_1 and b_2 respectively.

(b) Determine the $\mathcal{B}$-coordinate vector $[x]_{\mathcal{B}}$ of the vector $x = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}$.

Solution: Want to write:

$$x = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ 1 & 5 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Thus we are solving the system $\begin{bmatrix} -3 & 7 \\ 1 & 5 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}$. Thus reducing the augmented matrix we see:

$$\begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0. \end{bmatrix}$$

Hence $[x]_{\mathcal{B}} = \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}$.

6. Extra Credit: Suppose $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the linear transformation with standard matrix:

$$F = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

(a) (3 points) Determine F^{-1}

(b) (2 points) Using F^{-1}, determine a vector x such that $T(x) = \begin{bmatrix} 9 \\ -3 \\ -12 \end{bmatrix}$.