

## Ranks

1. Determine the largest possible rank and the smallest possible dimension for the null space given a matrix  $A$  of the following dimensions.
  - (a)  $6 \times 8$
  - (b)  $8 \times 6$
  - (c)  $1 \times 7$
  
2. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation.
  - (a) What is the dimension of the range of  $T$  if  $T$  is one-to-one? Explain.
  - (b) What is the dimension of the kernel of  $T$  if  $T$  is onto? Explain.
  
3. Suppose  $A_{m \times n}$  and  $B_{n \times p}$ .
  - (a) Prove  $\text{Rank}(AB) \leq \text{Rank}(A)$  by explaining why  $\text{Col}(AB)$  is a subspace of  $\text{Col}(A)$ .
  - (b) Prove  $\text{Rank}(AB) \leq \text{Rank}(B)$  using the above inequality and studying  $(AB)^T$ .
  
4. Prove the following theorem.

Let  $V$  be a  $p$ -dimensional vector space with  $p \geq 1$ . Then any linearly independent set of exactly  $p$  elements of  $V$  is in fact a basis for  $V$ . In addition any set of  $p$  elements of  $V$  that spans  $V$  is in fact a basis of  $V$ .
  
5. Prove that a matrix  $A_{m \times n}$  with  $m > n$  attains its maximum rank if and only if the columns of  $A$  are linearly independent.