Math 533 Exam

December 16 2015

Do at least **three** of the following five questions. If you complete more than three questions, indicate clearly which three you would like to be graded.

(1) Let $\Omega \subset \mathbb{R}^n$ be open. Suppose $u \in C(\Omega)$ has the property that

$$u(x) = \fint_{\partial B(x,r)} u\, dS$$

for any $B(x,r) \subset \Omega$. Show that u is smooth. You may need to use the fact that for any $\varepsilon > 0$, there exists a smooth function η_ε which is radial and supported in the ball of radius ε.

(2) Let Φ be the fundamental solution for the Laplacian on $\mathbb{R}^3$:

$$\Phi(x) = \frac{1}{4\pi|x|} \text{ for } x \neq 0.$$

Suppose u is harmonic on a smooth bounded domain $U \subset \mathbb{R}^3$. Show that for $x \in U$,

$$u(x) = \int_{\partial U} [\Phi(y-x)\partial_\nu u(y) - u(y)\partial_\nu \Phi(y-x)] dS_y.$$

(3) Let $U \subset \mathbb{R}^n$ be a smooth bounded domain. Show that the equation

$$\begin{aligned} u_{tt} - \triangle u &= f \text{ in } U_T \\ u &= g \text{ on } \Gamma_T \\ u_t &= h \text{ on } U \times \{t = 0\} \end{aligned}$$

has at most one solution $u \in C^2(\overline{U_T})$.

(4) Suppose u is harmonic and nonnegative on $B(0,2)$. Show that there exists $C > 0$ depending on the dimension such that for any $x, y \in B(0,1)$,

$$\frac{1}{C}u(y) \geq u(x) \geq Cu(y).$$

(5) Suppose $f \in C^\infty(\mathbb{R})$. Show that the equation

$$\begin{aligned} u_t + uu_x &= 0 \text{ for } t > 0 \\ u(x,0) &= f(x). \end{aligned}$$

has a solution which is continuous on $\{t \geq 0\}$ if f is nondecreasing. By means of a counterexample, show that this is not necessarily true if f is decreasing.