Math 533 Midterm

October 23, 2015.

Do at least **three** of the following five questions. If you complete more than three questions, your grade will be based on the best three.

(1) Define $u \in C^2(\Omega)$ to be subharmonic in Ω if

$$\triangle u \geq 0 \text{ in } \Omega.$$

Show that $u \in C^2(\Omega)$ is subharmonic in Ω if and only if

$$u(x) \leq \fint_{\partial B(x,r)} u \, dS$$

for every $B(x,r) \subset \Omega$.

(2) Suppose $u \in C^2(\overline{\Omega})$ solves

$$\begin{aligned} \triangle u &= u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Show that $u \equiv 0$ in Ω.

(3) Let $\Phi(x,t)$ be the fundamental solution to the heat equation:

$$\Phi(x,t) = \begin{cases} (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $f \in C^\infty(\mathbb{R}^n \times (0,\infty))$ has compact support, and let

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s) \, dy \, ds.$$

Assuming that u is smooth, show that

$$u_t - \triangle u = f$$

on $\mathbb{R}^n \times (0,\infty)$. You may need the fact that

$$\int_{\mathbb{R}^n} \Phi(x,t) dx = 1$$

for $t > 0$, and the fact that

$$\lim_{s \to 0^+} \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s) \, dy = f(x,t)$$

for $t > 0$.

(4) Let $g \in C(\mathbb{R}^n)$ be compactly supported, and $T > 0$. Show that the equation

$$\begin{aligned} u_t - \triangle u &= 0 \text{ in } \mathbb{R}^n \times (0, T) \\ u &= g \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{aligned}$$

has at most one solution $u \in C^2(\mathbb{R}^n \times [0, T])$ that satisfies the condition

$$\lim_{|x| \to \infty} u(x, t) = 0$$

for each $t \in (0, T)$.

(5) Suppose $u \in C^2(\mathbb{R} \times \mathbb{R})$ satisfies the equation

$$u_{tt} - 3u_{tx} + 2u_{xx} = 0$$

on $\mathbb{R} \times \mathbb{R}$. Show that if $u(x, 0) = u_t(x, 0) = 0$ for all $x > 0$, then $u(x, t) = 0$ for all $x, t > 0$. (Hint: this is the wave equation question.)