Problem Set 1

(1) Read Chapter 1 and section 2.1 of Evans. Also read section 2.2 up until the end of the proof of Theorem 4 (page 27).

(2) Do problem 1 and 2 from section 2.5 of Evans.

(3) Solve the equation $u_t + xu_x = 0$, with the initial condition $u(x, 0) = g(x)$, where g is C^1.

(4) ("Weak solutions")
i) Suppose $u \in C^1(\mathbb{R}^2)$ solves the transport equation

$$u_t + bu_x = 0$$

with initial condition $u(x, 0) = g(x)$. Show that if $v \in C^1(\mathbb{R}^2)$ satisfies $v(x, 0) = g(x)$ and

$$\int_{\mathbb{R}^2} v(\varphi_t + b\varphi_x) = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$, then $v = u$.

ii) Inspired by part i), let's define u to be a "weak solution" of $u_t + bu_x = 0$ if

$$\int_{\mathbb{R}^2} u(\varphi_t + b\varphi_x) = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. (I've put "weak solution" in quotes because this isn't quite the definition we'll use later.) Show that $u(x, t) = |x - bt|$ is a "weak solution" to $u_t + bu_x = 0$, even though it isn't differentiable.

iii) (Optional) Is $u(x, t) = |x - bt|$ the only continuous "weak solution" to $u_t + bu_x = 0$ with $u(x, 0) = |x|$? Note that this is hard (I think it is hard, anyway) and this is one of the reasons we should worry about better definitions.