Problem Set 10

(1) Reading: Read the beginning of section 4.1, up to the end of example 2 in 4.1.1 (page 171).

(2) Let $m, n \in \mathbb{Z}$. Define the Kronecker delta, δ_{mn}, to be 1 when $m = n$ and zero otherwise. Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx = \delta_{mn}.$$

You can interpret this from a linear algebra point of view: if we define the inner product of two functions f, g on the interval $[-\pi, \pi]$ by the integral

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g}(x)dx,$$

then what we have just shown is that the set of all $e^{inx}, n \in \mathbb{Z}$, forms an orthonormal set.

(3) Using the identity $e^{ix} = \cos x + i \sin x$, conclude that

$$\frac{2}{\pi}\int_{0}^{\pi} \sin(mx)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} \cos(mx)\cos(nx)dx = \delta_{mn}$$

for all $m, n \neq 0 \in \mathbb{Z}$.

(4) Suppose that $f(x) = \sum_{n=-N}^{N} a_n e^{inx}$, where $a_n \in \mathbb{C}$. Show that

$$\|f\|^2_{L^2([-\pi,\pi])} = 2\pi \sum_{n=-N}^{N} |a_n|^2,$$

and moreover, that

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

(5) Now suppose $f \in L^1([-\pi, \pi])$. Define

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

and show that

$$\lim_{n\to\pm\infty} a_n = 0.$$

(Hint: first do this for $f \in C^1$, and then use the fact that for every L^1 function f and every $\varepsilon > 0$, there is a C^1 function g such that $\|f - g\|_{L^1} < \varepsilon$. This result is sometimes called the Riemann-Lebesgue lemma.)

(6) Suppose $f \in C^1([-\pi, \pi])$, and define a_n as in the previous question. Define

$$S_N(x) = \sum_{n=-N}^{N} a_n e^{inx}.$$

Show that for each $x \in (-\pi, \pi)$,

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x - y) f(y) dy$$

where

$$K_N(\theta) = \frac{\sin((N + \frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$$

Hint: Use Fubini, and sum a geometric series.

(7) Show that K_N is continuous, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\theta) d\theta = 1.$$

(Hint: use the series form of K_N.) Now show that for $f \in C^1([-\pi, \pi])$,

$$\begin{aligned} S_N(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\theta)[\tilde{f}(x + \theta) - \tilde{f}(x)] d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin((N + 1/2)(\theta)) \left[\frac{\tilde{f}(x + \theta) - \tilde{f}(x)}{\sin(\frac{1}{2}\theta)} \right] d\theta \end{aligned}$$

for each $x \in (-\pi, \pi)$. Here $\tilde{f}$ is the periodic extension of f to the whole real line.

(8) Show that for $f \in C^1([-\pi, \pi])$, and each $x \in (-\pi, \pi)$

$$\lim_{N \to \infty} (S_N(x) - f(x)) = 0.$$

Moreover, show that if the periodic extension of f is C^1, then

$$\lim_{N \to \infty} (S_N(x) - f(x)) = 0$$

for all $x \in [-\pi, \pi]$. Hint: think about your reasoning for question 5.

(9) Now show that for $f \in L^2([-\pi, \pi])$,

$$\lim_{N \to \infty} \|S_N - f\|_{L^2([\pi,-\pi])} = 0.$$

You will need to use the fact that for every $\varepsilon > 0$ and every $f \in L^2([-\pi, \pi])$, there exists a periodic $g \in C^1(\mathbb{R})$ such that $\|f - g\|_{L^2([-\pi,\pi])} < \varepsilon$.

We can summarize what we have just shown by saying that any f in $L^2([-\pi, \pi])$ can be written as a sum of the form

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

in the L^2 sense. The linear algebra interpretation is that the set $\{e^{inx}\}_{n \in \mathbb{Z}}$ can be viewed as a basis for $L^2([-\pi, \pi])$. (!) Because of your work in question 3, you should be convinced that something similar is true for the set of all $\cos(mx)$ and the set of all $\sin(nx)$ on the interval $[0, \pi]$.