Problem Set 11

(1) Reading: Read section 4.3 up to the end of Example 4 (p187-194).

(2) Suppose $f \in L^1(\mathbb{R}^n)$, and $\alpha \in \mathbb{R}$. Show that

$$\text{if } g(x) = f(x-\alpha) \text{ then } \hat{g}(\xi) = \hat{f}(\xi)e^{-i\alpha\cdot\xi}$$

and

$$\text{if } g(x) = f(x)e^{i\alpha\cdot x} \text{ then } \hat{g}(\xi) = \hat{f}(\xi-\alpha).$$

(3) Suppose $f, g \in L^1(\mathbb{R}^n)$. Show that

$$\widehat{f * g}(\xi) = (2\pi)^{n/2}\hat{g}(\xi)\hat{f}(\xi).$$

(4) Suppose $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Show that

$$\text{if } g(x) = \overline{f(-x)} \text{ then } \hat{g}(\xi) = \overline{\hat{f}(\xi)}$$

and

$$\text{if } g(x) = f(x/\lambda) \text{ then } \hat{g}(\xi) = \lambda^n\hat{f}(\lambda x).$$

(5) Suppose $f \in L^1(\mathbb{R}^n)$. Show that $\hat{f}$ is continuous. (Use question 2 together with the dominated convergence theorem).

(6) Suppose $f \in C^1(\mathbb{R})$ is compactly supported. Show that

$$\lim_{\xi\to\pm\infty} \hat{f}(\xi) = 0.$$

(7) Suppose $f \in L^1(\mathbb{R})$. Show that

$$\lim_{\xi\to\pm\infty} \hat{f}(\xi) = 0.$$

This is sometimes called the Riemann-Lebesgue lemma. Hint: use the previous question, together with the fact that for every $f \in L^1(\mathbb{R})$ and every $\varepsilon > 0$, there exists a compactly supported $g \in C^1(\mathbb{R})$ such that

$$\|f-g\|_{L^1(\mathbb{R})} < \varepsilon.$$