Problem Set 12

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = e^{-a|x|}$$

where $a > 0$. Find $\hat{f}$.

(2) Suppose $f \in C_0^2(\mathbb{R})$, and suppose $u = u(x, y)$ has a Fourier transform in x which is bounded on the upper half plane $\mathbb{R}^2_+ = \{y > 0\}$ and satisfies

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2_+ \\ u(x,0) &= f(x). \end{aligned}$$

By taking the Fourier transform in the x variable and solving the resulting ODE, find an explicit formula for u in terms of f. Compare this to the formula we obtained in the section on Green's functions.

(3) Suppose f and u are as in the previous question. Find a formula for $u_y(x, 0)$. (It suffices to give a formula for $\hat{u}_y(\xi, 0)$ in terms of $\hat{f}$, where the hat indicates the Fourier transform in the x variable.) The map $\Lambda : f \to u_y(x, 0)$ is sometimes called the Dirichlet-to-Neumann map: it maps the Dirichlet boundary data for the problem to the Neumann boundary data. What is the operator Λ^2 – i.e., what is $\Lambda(\Lambda(f))$?

(4) Suppose $g, h \in C_0^2(\mathbb{R})$, and suppose $u = u(x, t)$ is locally bounded on the upper half plane $\mathbb{R}^2_+ = \{t > 0\}$ and satisfies the wave equation

$$\begin{aligned} (\partial_t^2 - \partial_x^2)u &= 0 \text{ in } \mathbb{R}^2_+ \\ u(x,0) &= g(x). \\ u_t(x,0) &= 0. \end{aligned}$$

By taking the Fourier transform in the x variable and solving the resulting ODE, find an explicit formula for u in terms of g. Compare this to d'Alembert's formula.