Problem Set 3

(1) Reading: Read Section 2.2 from the note on positivity after the proof of Theorem 4 (p27) to the end of the proof of Theorem 13 (page 36).

(2) Do problem 3,5, and 6 from section 2.5 of Evans.

(3) (Interior gradient estimate): Show that there exists a constant c depending only on the dimension such that

$$\sup_{B(0,1/2)} |\nabla u| \leq c \sup_{\partial B(0,1)} |u|$$

whenever u is harmonic in $B(0,1)$. Hint: Consider a function of the form $\eta^2|\nabla u|^2 + au^2$, where a is constant, and $\eta \in C_0^2(B(0,1))$ with $\eta \equiv 1$ in $B(0,1/2)$. Use question 5 from Evans.

(4) Use the previous question to show that for each $\alpha \in [0,1]$, there exists a constant c_α such that if u is harmonic in $B(0,1)$, then

$$|u(x) - u(y)| \leq c_\alpha |x-y|^\alpha \sup_{\partial B(0,1)} |u|$$

whenever $x, y \in B(0,1/2)$.

(5) (Cacciopoli inequality): Suppose u is harmonic in Ω. Show that if $\eta \in C_0^1(\Omega)$, then

$$\int_\Omega \eta^2 |\nabla u|^2 \leq C \int_\Omega |\nabla \eta|^2 u^2$$

where C depends only on Ω.

(6) Suppose u is harmonic in $B(0,1)$. Using the Cacciopoli inequality, show that if $0 \leq r < R \leq 1$, then

$$\int_{B(0,r)} |\nabla u|^2 \leq \frac{C}{(R-r)^2} \int_{B(0,R)} u^2$$

for some constant C.

(7) Suppose $0 < R \leq 1$. Using the Cacciopoli inequality, show that there exists $\theta \in (0,1)$ such that

$$\int_{B(0,R/2)} u^2 \leq \theta \int_{B(0,R)} u^2$$

whenever u is harmonic in $B(0,1)$.

For this question you may need to use the Poincaré inequality: if Ω is a smooth bounded domain, and $v \in C^1(\Omega)$ and $v = 0$ on $\partial\Omega$, then there exists a constant C depending only on Ω such that

$$\int_\Omega |v|^2 \leq C \int_\Omega |\nabla v|^2.$$

For the remaining questions, you will need the following definition.

Definition 0.1. *Let $a_{ij}, c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ be constant. An operator L of the form*

$$Lu = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + b \cdot \nabla u + cu$$

is called elliptic if there exists $\lambda > 0$ such that

$$\xi \cdot A\xi > \lambda|\xi|^2$$

for any $\xi \in \mathbb{R}^n$, where A is the matrix with entries a_{ij}.

You should check that the Laplacian is an elliptic operator.

(8) Prove the maximum principle for an elliptic operator with $c = 0$.

(9) (optional) Prove the strong maximum principle for an elliptic operator with $c = 0$. (Note: this is much harder. Why?)