Problem Set 4

(1) Reading: Read Section 2.2 from wherever you left off until the end of the section.

(2) Do problem 7, 8, and 9 from section 2.5 of Evans.

For the remaining problems you will need the following definition. Define $N(x, y)$ to be a Neumann function for Ω if $N(x, y)$ takes the form

$$N(x, y) = \Phi(y - x) - \psi^x(y)$$

where Φ is the fundamental solution to the Laplacian, and for each $x \in \Omega$, $\psi^x \in C^2(\Omega)$ solves

$$\begin{aligned} \triangle \psi^x &= 0 \text{ on } \Omega \\ \partial_\nu \psi^x(y) &= \partial_\nu \Phi(y - x) \text{ on } \partial\Omega. \end{aligned}$$

(3) Prove that if a Neumann function for Ω exists, then it is not unique.

(4) Prove that if a Neumann function $N(x, y)$ for Ω exists, then any solution to

$$\begin{aligned} \triangle u &= f \text{ on } \Omega \\ \partial_\nu u &= g \text{ on } \partial\Omega \end{aligned}$$

has the form

$$u(x) = \int_\Omega N(x, y) f(y) dy + \int_{\partial\Omega} N(x, y) g(y) dS_y.$$

(5) Using your result in the previous two questions, show that if a Neumann function for Ω exists, then

$$\begin{aligned} \triangle u &= f \text{ on } \Omega \\ \partial_\nu u &= g \text{ on } \partial\Omega \end{aligned}$$

has a solution only if

$$\int_\Omega f + \int_{\partial\Omega} g = 0.$$

(6) Find a Neumann function for the half space $\mathbb{R}^n_+$.