# Problem Set 6

(1) Reading: Read Section 2.3 from the proof of Theorem 8 (p 59) to the end. Also read Section 2.4 until the end of Theorem 1 (page 68).

(2) Do problems 15 and 18 from section 2.5 of Evans.

The remaining questions are about energy methods. In particular the proofs of Theorem 16 in section 2.2 and Theorem 10 in section 2.3 may be helpful.

(3) Let $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Show that if $u_1, u_2 \in C^2(\overline{\Omega})$ are two solutions of the problem

$$\begin{aligned} \triangle u &= f \text{ in } \Omega \\ \partial_\nu u &= g \text{ on } \partial\Omega \end{aligned}$$

then $u_1 - u_2$ is constant on each connected component of $\Omega$.

(4) Suppose $f \in C(U_T)$, $g \in C(\partial U \times (0, T])$, and $h \in C(U)$, and consider the problem

$$\begin{aligned} u_t - \triangle u &= f \text{ in } U_T \\ \partial_\nu u &= g \text{ on } \partial U \times (0, T] \\ u &= h \text{ on } U \times \{0\} \end{aligned}$$

Show that this problem has at most one solution $u \in C_1^2(\overline{U}_T)$.

(5) Suppose $h \in C(U)$, and $u \in C_1^2(\overline{U}_T)$ solves

$$\begin{aligned} u_t - \triangle u &= 0 \text{ in } U \times (0, \infty) \\ \partial_\nu u &= 0 \text{ on } \partial U \times (0, \infty) \\ u &= h \text{ on } U \times \{0\}. \end{aligned}$$

Show that

$$\lim_{t \to \infty} u(x, t) = \frac{1}{|U|} \int_U h(x) dx.$$

In particular, $u(x, t)$ converges to a constant.

(6) Suppose $h \in C(U)$, and $u \in C_1^2(\overline{U}_T)$ solves

$$\begin{aligned} u_t - \triangle u &= 0 \text{ in } U \times (0, \infty) \\ u &= 0 \text{ on } \partial U \times (0, \infty) \\ u &= h \text{ on } U \times \{0\}. \end{aligned}$$

Show that

$$\lim_{t \to \infty} u(x,t) = 0.$$

(7) Suppose $f, h \in C(U)$ and $g \in C(\partial U)$. Suppose $u \in C_1^2(\overline{U}_T)$ solves

$$\begin{aligned} u_t - \triangle u &= f \text{ in } U \times (0, \infty) \\ u &= g \text{ on } \partial U \times (0, \infty) \\ u &= h \text{ on } U \times \{0\} \end{aligned}$$

and $v \in C^2(\overline{\Omega})$ solves

$$\begin{aligned} -\triangle v &= f \text{ in } U \\ v &= g \text{ on } \partial U \end{aligned}$$

Show that

$$\lim_{t \to \infty} u(x,t) = v(x).$$

Note in particular that solutions to the homogeneous heat equation converge to harmonic functions.