Problem Set 7

(1) Reading: Read Section 2.4 from Theorem 1 (p 68) to the end.

(2) Do problems 19, 21b), and 22 from section 2.5 of Evans.

I would have you do 21a) too, but Evans leaves out the best part.

(3) Assume the electric field $E = (E^1, E^2, E^3)$ and the magnetic field $B = (B^1, B^2, B^3)$ solve Maxwell's equations in free space:

$$\begin{aligned} \mu_0 \varepsilon_0 E_t &= \nabla \times B \\ B_t &= -\nabla \times E \\ \nabla \cdot E &= \nabla \cdot B = 0. \end{aligned}$$

Here μ_0 and ε_0 are constants: they are the magnetic permeability and electrical permittivity of free space, respectively. Show that

$$E_{tt} - \frac{1}{\mu_0 \varepsilon_0} \triangle E = 0, \quad B_{tt} - \frac{1}{\mu_0 \varepsilon_0} \triangle B = 0.$$

Note that this implies that electromagnetic fields propagate as waves that travel at speed

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

When J.C. Maxwell discovered this in the 1800s, he had laboratory measurements of μ_0 and ε_0 that allowed him to calculate

$$c \simeq 3 \times 10^8 m/s.$$

This is remarkably close to the speed of light. This led Maxwell to make the astonishing conjecture, later confirmed, that *light is an electromagnetic wave*. This discovery of the structure of the wave equation in the fundamental equations of electricity and magnetism is at the heart of one of the most influential scientific discoveries of the 19th century.

(4) Suppose $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Find an explicit formula for the solution of

$$\begin{aligned} u_{tt} - 2u_{tx} - 3u_{xx} &= 0 \text{ on } \mathbb{R} \times \mathbb{R} \\ u|_{t=0} = g, \quad u_t|_{t=0} &= h \end{aligned}$$

in terms of g and h.

(5) Suppose $f \in C^2(\mathbb{R})$. Find an explicit formula for the solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \text{ on } \mathbb{R}_+ \times \mathbb{R}_+ \\ u|_{t=0} = u_t|_{t=0} &= 0 \end{aligned}$$

with the boundary condition

$$u|_{x=0} = f(t).$$