Math 533 Sample Exam

Do at least **three** of the following five questions. If you complete more than three questions, your grade will be based on the best three.

(1) Let B_1 denote the open unit ball in $\mathbb{R}^n$, centred at the origin. Let $f \in C(B_1)$ such that $|f(x)| \leq 1$ for $x \in B_1$. Assume $u \in C^2(\overline{B}_1)$ solves

$$\begin{aligned} \triangle u &= f \text{ in } B_1 \\ u &= 0 \text{ on } \partial B_1 \end{aligned}$$

Prove that

$$|u(x)| \leq \frac{1}{2n}(1 - |x|^2).$$

(2) Let $\Phi(x)$ be the fundamental solution to the Laplacian on $\mathbb{R}^2$:

$$\Phi(x) = -\frac{1}{2\pi} \log |x|$$

for $x \neq 0$. Let f be a smooth compactly supported function on $\mathbb{R}^n$, and let

$$J_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \triangle_y f(x-y) dy.$$

Show that

$$\lim_{\varepsilon \to 0} J_\varepsilon = -f(x).$$

(3) Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Show that the equation

$$\begin{aligned} u_t - \triangle u &= f \text{ in } \Omega \times \{t > 0\} \\ u(x,0) &= g \text{ for } x \in \Omega \\ \partial_\nu u &= h \text{ on } \partial\Omega \times \{t > 0\} \end{aligned}$$

has at most one solution $u \in C^2(\overline{\Omega} \times \{t \geq 0\})$.

(4) Let $B(0,1)$ be the ball of radius 1 in $\mathbb{R}^3$ centred at the origin. Suppose $u \in C^2(\mathbb{R}^3 \times \{t \geq 0\})$ solves the wave equation

$$u_{tt} - \triangle u = v_{tt} - \triangle v = 0 \text{ on } \mathbb{R}^3 \times \{t > 0\}$$

with

$$u_t(x,0) = 0 \text{ for all } x \in \mathbb{R}^3$$

and

$$u(x,0) = 0 \text{ for all } x \in \mathbb{R}^3 \setminus B(0,1).$$

Show that there exists some $T > 0$ such that if $u(x,t) = 0$ for all $x \in \mathbb{R}^3 \setminus B(0,1)$ and $0 \le t \le T$, then $u \equiv 0$ on $\mathbb{R}^3 \times \{t \ge 0\}$.

(5) Suppose $u(x,y)$ solves

$$\begin{aligned} u_x u_y &= u \text{ for } x > 0 \\ u(0,y) &= y^2. \end{aligned}$$

Solve for u.