1. Continuity and Limits: Definitions

In the notes on the real numbers we defined a continuous function as follows.

Definition 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for every open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is open.*

Theorem 4.3 from those notes provides an equivalent definition:

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for every $p \in \mathbb{R}$ and every open interval I_1 which contains $f(p)$, there exists an open interval I_2 which contains p such that $f(I_2) \subseteq I_1$.*

Replacing open intervals by open balls and rephrasing in terms of inequalities gives the following definition:

Definition 3. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for every $p \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\text{if } |x - p| < \delta \text{ then } |f(x) - f(p)| < \varepsilon.$$

Theorem 1.1. *The above three definitions are all equivalent.*

Exercise 1.2. *Use Definition 3 to show that $1, x$, and x^2 are continuous. Show that for any positive real number q, there exists a unique positive number p such that $p^2 = q$.*

Definition 4. *Let $f : \mathbb{R} \to \mathbb{R}$, and $p \in \mathbb{R}$. We say that $\lim_{x \to p} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\text{if } 0 < |x - p| < \delta \text{ then } |f(x) - L| < \varepsilon.$$

The limit is well defined:

Theorem 1.3. *Show that if the limit exists then it is unique: i.e. if*

$$\lim_{x \to p} f(x) = L \text{ and } \lim_{x \to p} f(x) = M$$

then $L = M$.

Now we have a fourth definition of continuity!

Definition 5. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at p if*

$$\lim_{x \to p} f(x) = f(p).$$

We say f is continuous if this holds for every $p \in \mathbb{R}$.

Examination of the definition of limit shows that only the values of $f(x)$ for x near (and not equal to) p determine the limit at p. This allows us to define the limit for functions whose domain is not all of $\mathbb{R}$.

Definition 6. *Suppose $A \subset \mathbb{R}$ is open, and $p \in A$, and suppose $f : B \to \mathbb{R}$ where $A \setminus \{p\} \subseteq B$.*

We say that $\lim_{x \to p} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } 0 < |x - p| < \delta \text{ and } x \in B \text{ then } |f(x) - L| < \varepsilon.$$

Remark: In the spirit of the $\varepsilon - \delta$ definitions of $\lim_{x\to a} f(x) = L$, we can also give definitions for

$$\lim_{x\to\infty} f(x) = L, \qquad \lim_{x\to a} f(x) = \infty,$$

and several other variations.

2. Limit Theorems

Theorem 2.1. *Suppose*

$$\lim_{x\to p} f(x) = L \quad \text{and} \quad \lim_{x\to p} g(x) = M.$$

Then

$$\begin{aligned} &\lim_{x\to p}(f(x)+g(x)) &&= L + M, \\ &\lim_{x\to p} f(x)g(x) &&= LM, \\ \text{and} \quad &\lim_{x\to p} \frac{1}{g(x)} &&= \tfrac{1}{M} \quad \text{if } M \neq 0. \end{aligned}$$

Remark: This shows that the sum, product, and quotient of continuous functions is continuous (as long as the denominator is nonzero, in the case of the quotient).

Theorem 2.2 (Squeeze Theorem*)**.** *Suppose* $f(x) \le g(x) \le h(x)$ *for all* $x \in \mathbb{R}$, *and*

$$\lim_{x\to p} f(x) = \lim_{x\to p} h(x) = L.$$

Then $\lim_{x\to p} g(x) = L$.

Theorem 2.3. *Suppose* $\lim_{x\to p} g(x) = L$, *and* f *is continuous at* L. *Then*

$$\lim_{x\to p} f(g(x)) = f(L).$$

3. Derivatives

Definition 7. *Suppose $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. We say f is differentiable at a if the limit*

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Then we say that this limit is the derivative of f at a, and denote this by $f'(a)$.

Theorem 3.1. *If f is differentiable at a, then f is continuous at a.*

Note that the converse is false: find a counterexample!

A straightforward application of the definition shows that the identity function $f(x) = x$ and the constant function $g(x) = c$ are differentiable everywhere, and their derivatives are 1 and 0, respectively, for all x.

Now the following rules for sums, products, and quotients let us differentiate any rational function.

Theorem 3.2 (Sum, Product, and Quotient Rules)**.** *If f and g are differentiable at x, then*

- *$f + g$ is differentiable at x and $(f + g)'(x) = f'(x) + g'(x)$.*
- *$f \cdot g$ is differentiable at x, and $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.*
- *If $g'(x) \neq 0$ then $1/g$ is differentiable at x, and $(1/g)'(x) = -g'(x)/g^2(x)$.*

Finally, we prove the chain rule. First a lemma.

Lemma 3.3. *Suppose g is differentiable at a and f is differentiable at $g(a)$. Define*

$$\phi(h) = \begin{cases} \frac{f(g(a+h))-f(g(a))}{f'(g(a))} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{otherwise} \end{cases}$$

Show that ϕ is continuous at 0.

Theorem 3.4 (Chain Rule)**.** *Suppose g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a, and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Definition 8. *Let A be a subset of the domain of f. We say that $x \in A$ is a maximum point for f on A if $f(x) \geq f(y)$ for all $y \in A$. A similar definition holds for minimum point.*

Theorem 3.5. *Suppose f is defined on (a, b), and x is a maximum or minimum point of f on (a, b). If f is differentiable at x then $f'(x) = 0$.*

Lemma 3.6 (Rolle's Theorem*)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.*

Theorem 3.7 (Mean Value Theorem*)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x \in (a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 3.8. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).*

Corollary 3.9 (Cauchy Mean Value Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x \in (a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 3.10 (L'Hôpital's Rule)**.** *Suppose $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0$ and $\lim_{x \to p} \frac{f'(x)}{g'(x)}$ exists. Then*

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)}.$$

In particular the limit on the left exists.

4. Uniform Continuity

Definition 9. *We say f is uniformly continuous on A if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, p \in A$,*

$$\text{if } |x - p| < \delta \text{ then } |f(x) - f(p)| < \varepsilon.$$

Exercise 4.1. *Give an example of a function that is continuous on $(0, 1)$ but not uniformly continuous there.*

Theorem 4.2 (*)**.** *If A is compact and f is continuous on A then f is uniformly continuous on A.*

5. Integrals

Definition 10. *A partition P of $[a,b]$ is a finite collection of points $t_0, \ldots, t_n$ such that*

$$a = t_0 < \ldots < t_n = b.$$

Definition 11. *Suppose f is bounded on $[a,b]$, and P is a partition of $[a,b]$. The upper and lower sums of f on $[a,b]$ with respect to P are the quantities*

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{and} \quad L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

where M_i and m_i are the supremum and infimum, respectively, of the sets

$$\{f(x) | x \in [t_{i-1}, t_i]\}.$$

Lemma 5.1. *Let P and Q be partitions of $[a,b]$ such that all the points of P are also in Q. Then*

$$L(f,P) \leq L(f,Q) \quad \text{and} \quad U(f,P) \geq U(f,Q).$$

Theorem 5.2. *Let P_1, P_2 be partitions of $[a,b]$. Then*

$$L(f,P_1) \leq U(f,P_2).$$

Definition 12. *Suppose f is bounded on $[a,b]$. We say f is (Riemann) integrable on $[a,b]$ if*

$$\inf_P U(f,P) = \sup_P L(f,P),$$

and define this quantity to be the integral

$$\int_a^b f(x)dx.$$

Exercise 5.3. *Show that $f(x) = x^2$ is integrable on $[0,1]$. Find a function which is not integrable on $[0,1]$.*

Lemma 5.4. *Suppose f is bounded on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if for all $\varepsilon > 4$, there exists a partition P of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon.$$

Theorem 5.5. *Suppose f is continuous on $[a,b]$. Then f is integrable on $[a,b]$.*

Theorem 5.6. *Suppose f is integrable on $[a,b]$ and on $[b,c]$. Then f is integrable on $[a,c]$, and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

Remark: This formula justifies the definition

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Moreover, it follows from this theorem that not all integrable functions are continuous. (Why?)

Theorem 5.7. *Suppose f is integrable on $[a, b]$, and c is constant. Then cf is integrable on $[a, b]$, and*

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

Theorem 5.8. *Suppose f, g are integrable on $[a, b]$. Then $f + g$ is integrable on $[a, b]$, and*

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Lemma 5.9. *Suppose $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Theorem 5.10 (1st Fundamental Theorem*)**.** *Suppose f is integrable on $[a, b]$ and define*

$$F(x) = \int_a^x f(t)dt.$$

If f is continuous at c then F is differentiable at c and

$$F'(c) = f(c).$$

Theorem 5.11 (2nd Fundamental Theorem*)**.** *Suppose f is integrable on $[a, b]$ and $f = g'$ for some g. Then*

$$\int_a^b f(x)dx = g(b) - g(a).$$