1. Problem Set 1

Recall from class the following definition:

Definition 1 (Definition of $\mathbb{N}$)**.** *The natural numbers are a set $\mathbb{N}$, together with a successor function $S : \mathbb{N} \to \mathbb{N}$, which satisfies the following axioms:*

Axiom 1: *: $\mathbb{N}$ is nonempty, and contains an object called 0.*
Axiom 2: *: If $n, m \in \mathbb{N}$, then $S(n) \neq S(m)$.*
Axiom 3: *: There is no $n \in \mathbb{N}$ such that $0 = S(n)$.*
Axiom 4: *: If $A \subset \mathbb{N}$, and $0 \in A$, and for every $a \in A$, $S(a) \in A$, then $A = \mathbb{N}$.*

Definition 2. *Suppose $m \in \mathbb{N}$. We define $0 + m = m$. Moreover if we have defined $n + m$, then we define $S(n) + m = S(n + m)$.*

Problem 1 (Commutativity of Addition)**.** *Suppose $n, m \in \mathbb{N}$. Recall from class that we showed that $n + 0 = n$.*

- *Show that $n + S(m) = S(n + m)$.*
- *Show that $n + m = m + n$.*

Problem 2 (Associativity of Addition – optional)**.** *Suppose $n, m, p \in \mathbb{N}$. Then $n + (m + p) = (n + m) + p$.*

Problem 3 (Cancellation Law)**.** *Suppose $n, m, p \in \mathbb{N}$, and $n + m = n + p$. Show that $m = p$.*

Problem 4. *Using our definition of addition as a model, define the product nm of two natural numbers n and m.*

Problem 5. *Show, using your definition of multiplication, that $nm = mn$ for all natural numbers m and n.*

Definition 3. *Let $n, m \in \mathbb{N}$. We define $n \leq m$ if $m = n + p$ for some $p \in \mathbb{N}$, and $n < m$ if $n \leq m$ and $n \neq m$.*

Problem 6. *Show that if $n, m \in \mathbb{N}$ and $n \neq m$ then $n < m$ or $m < n$.*

Problem 7. *Show that if $a, b, c \in \mathbb{N}$ and $a < b$ and $b < c$ then $a < c$.*

Problem 8. *Suppose $n, m \in \mathbb{N}$ and $n < m$. Show that the statement $m < n$ is false.*

Definition 4. *Let $A \subset \mathbb{N}$. We say that a is a first point of A if $a \in A$, and $a \leq b$ for all $b \in A$. We say that a is a last point of A if $a \in A$ and $b \leq a$ for all $b \in A$.*

Problem 9. *Show that 0 is a first point of $\mathbb{N}$.*

Problem 10. *Show that every subset of $\mathbb{N}$ has a first point.*

Problem 11. *Show that every finite subset of $\mathbb{N}$ has a last point. (Before you begin: define finite!)*

Problem 12. *Show that not every subset of $\mathbb{N}$ has a last point.*