1. Problem Set 10

Problem 1. *Suppose that f is continuous at a and there exists an open interval I containing a such that $f'(x)$ exists for all x in $I \setminus \{a\}$. Suppose also that $\lim_{x \to a} f'(x)$ exists. Show that $f'(a)$ exists and*

$$f'(a) = \lim_{x \to a} f'(x).$$

Note that this means the only way in which the derivative can fail to be continuous is that the limit at some point does not exist. Hint: what does L'Hôpital have to do with this?

Problem 2. *Suppose $f'(x) > 0$ for all $x \in (a, b)$. Show that f is increasing on (a, b). (What does increasing mean?)*

Problem 3. *Suppose $f'(x) = 0$. Show that f has a local minimum at x if $f''(x) > 0$, and moreover, that if x has a local minimum at x and $f''(x)$ exists then $f''(x) \geq 0$.*

Problem 4. *Suppose $f'(x) \geq 4$ for all $x \in [0, 1]$. Show there is an interval of length $1/4$ on which $|f(x)| > 1$.*

Problem 5. *Suppose f is twice differentiable on $(0, 1)$ and continuous on $[0, 1]$, and there exists a function g defined on $(0, 1)$ such that*

$$f''(x) + f'(x)g(x) - f(x) = 0$$

for all $x \in (0, 1)$. Show that if $f(0) = f(1) = 0$ then $f(x) = 0$ for all $x \in [0, 1]$.

The rest of this problem set explores inverse functions.

Definition 1. *We define a function $f : A \to B$ to be* one-to-one *if $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$.*

Definition 2. *Given a one-to-one function $f : A \to B$, we define $f^{-1} : f(A) \to A$ by $f^{-1}(f(a)) = a$.*

One needs to convince oneself that this is really a proper definition of a function.

Problem 6. *Suppose $f : (a, b) \to \mathbb{R}$ is continuous and one-to-one on (a, b). Show that f is either increasing or decreasing on (a, b).*

Problem 7. *Suppose $f : (a, b) \to \mathbb{R}$ is continuous and one-to-one on (a, b). Show that f^{-1} is continuous.*

Problem 8. *Suppose f is continuous and one-to-one on an open interval containing a, and $f(a) = b$. Suppose moreover that f is differentiable at a. Show that f^{-1} is differentiable at b if and only if $f'(a) \neq 0$, and*

$$f^{-1'}(b) = \frac{1}{f'(f^{-1}(b))}.$$

Convince yourself that you now know what $x^{1/n}$ means, where it is continuous and differentiable, and how to differentiate it!