1. Problem Set 4

All references to "the notes" refer to the notes on the real numbers posted on the course webpage. If a question asks you to prove a proposition from the notes, you may freely use any *previous* proposition, lemma, theorem, corollary, etc. from the notes in your proof.

Otherwise you can use anything from the notes in your proof. In addition, you can use the results of previous problems in subsequent problems.

Problem 1. *Suppose $A \subset \mathbb{R}$ is open, bounded, and connected. Show that A is an open interval.*

Problem 2. *Prove Theorem 4.3 of the notes.*

Problem 3. *Prove Lemma 4.4 of the notes. Find a counterexample to demonstrate that $f(X \cap Y)$ and $f(X) \cap f(Y)$ are not always equal.*

Problem 4. *Prove Theorem 4.5 of the notes.*

Problem 5. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(\mathbb{R})$ is finite. Show that $f(\mathbb{R})$ consists of exactly one point.*

Problem 6. *For $A \subset \mathbb{R}$, define the interior of A by*

$$\operatorname{int} A = \{x \in A| \text{ there exists an open interval } (a,b) \text{ such that } x \in (a,b) \subset A\}.$$

Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every set $A \subset \mathbb{R}$,

$$f^{-1}(\operatorname{int} A) \subset \operatorname{int} (f^{-1}(A)).$$

Problem 7. *Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every closed set $X \subset \mathbb{R}$, the preimage $f^{-1}(X)$ is closed.*

Problem 8. *Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every set $A \subset \mathbb{R}$,*

$$f(\overline{A}) \subset \overline{f(A)}.$$