1. Problem Set 6

All references to "the notes" refer to the notes on the real numbers posted on the course webpage. If a question asks you to prove a proposition from the notes, you may freely use any *previous* proposition, lemma, theorem, corollary, etc. from the notes in your proof.

Otherwise you can use anything from the notes in your proof. In addition, you can use the results of previous problems (even on previous problem sets) in subsequent problems.

Problem 1. *Prove Corollary 5.9 of the notes.*

Definition 1. *A* Dedekind cut *is a subset A of the rational numbers, with the properties that*

- *if $x \in A$, and $y < x$, then $y \in A$.*
- *A is bounded above.*
- *A has no last point.*

We denote by $\mathcal{D}$ the set of all Dedekind cuts.

Definition 2. *Suppose $A, B \in \mathcal{D}$. We define*

$$A < B \text{ if } A \subset B \text{ and } A \neq B.$$

Problem 2. *Show that $<$ is an ordering on $\mathcal{D}$.*

Problem 3. *Show that $\mathcal{D}$ has no first point or last point.*

Problem 4. *Suppose that $X \subset \mathcal{D}$ is nonempty and bounded above. Show that X has a supremum.*

Now we can define open sets on $\mathcal{D}$ by using the ordering $<$.

Problem 5. *Show that $\mathcal{D}$ is connected.*

By now the purpose of these exercises should be clear: we are going to prove that $\mathcal{D}$ satisfies the axioms of the real numbers!

Definition 3. *If $A, B \in \mathcal{D}$, define*

$$A + B = \{a + b | a \in A, b \in B\}.$$

Problem 6. *Prove that $\mathcal{D}$ satisfies field axioms 1-4. How are 0 and $-A$ defined?*

Definition 4. *If $A, B \in \mathcal{D}$, and $A, B \geq 0$, define*

$$A \cdot B = \{a \cdot b | a \in A, b \in B, a, b \geq 0\} \cup \{x \in \mathbb{Q} | x < 0\}.$$

If $A \geq 0$ and $B < 0$ we define $A \cdot B = -(A \cdot (-B))$; if $A < 0$ and $B \geq 0$ then we define $A \cdot B = -((-A) \cdot B)$; if $A, B < 0$ we define $A \cdot B = (-A) \cdot (-B)$.

Problem 7. *(Optional) Show that $\mathcal{D}$ satisfies field axioms 5-10.*

Problem 8. *Show that $\mathcal{D}$ is an ordered field.*