1. Problem Set 7

Problem 1. *Recall the set of Dedekind cuts $\mathcal{D}$ from the last problem set. Define*

$$\mathbb{Q}' = \{\{x \in \mathbb{Q} | x < q\} | q \in \mathbb{Q}\}.$$

Show that any open interval in $\mathcal{D}$ contains an element of $\mathbb{Q}'$. We say that this means $\mathbb{Q}'$ is dense *in $\mathcal{D}$.*

From now on we will take $\mathbb{R}$ to be $\mathcal{D}$, but we will mostly ignore the Dedekind-ness of $\mathbb{R}$. Note that what you've shown above is that $\mathbb{Q}$ is dense in $\mathbb{R}$.

In several references this is given in some way as an axiom of $\mathbb{R}$, but it actually follows from the previous axioms.

Problem 2. *Show that the sequence $\{1/n\}$ converges to 0. Do you need to use the density of $\mathbb{Q}$ in $\mathbb{R}$?*

Problem 3. *Show that every convergent sequence is bounded.*

Problem 4. *Suppose a sequence $\{a_n\}$ is bounded above and nondecreasing: $a_n \geq a_m$ whenever $n > m$. Show that $\{a_n\}$ converges.*

Definition 1. *Suppose $\{a_n\}$ is a bounded sequence. Define new sequences*

$$u_n = \sup\{a_m | m > n\} \quad \text{and} \quad l_n = \inf\{a_m | m > n\}$$

We say that

$$\limsup a_n = p \text{ if } \lim_{n\to\infty} u_n = p$$

and

$$\liminf a_n = p \text{ if } \lim_{n\to\infty} l_n = p.$$

Problem 5. *Suppose $\{a_n\}$ is a bounded sequence. Show that $\limsup a_n$ and $\liminf a_n$ always exist.*

Problem 6. *Suppose $\{a_n\}$ is a bounded sequence, and $\limsup a_n = \liminf a_n = p$. Show that $\{a_n\}$ converges to p.*

Problem 7. *Suppose $A \subset \mathbb{R}$. Show that $p \in \overline{A}$ if and only if there exists a sequence $\{a_n\}$ such that each $a_n \in A$, and $\{a_n\}$ converges to p.*