## 1. Problem Set 8

**Problem 1.** *Suppose $\{a_n\}$ is a sequence of real numbers. We say that $\{a_n\}$ is a fast Cauchy sequence if there exists $M \in \mathbb{R}$ such that for all $N \in \mathbb{N}$,*

$$M > \sum_{n=1}^{N} |a_n - a_{n+1}|.$$

*Show that if $\{a_n\}$ is a fast Cauchy sequence then it is Cauchy.*

**Problem 2.** *Show that if $\{a_n\}$ is Cauchy then it has a subsequence which is fast Cauchy.*

**Problem 3.** *Let $f : \mathbb{R} \to \mathbb{R}$. We say that $f$ is a contraction map if there exists $c \in (0, 1)$ such that for all $x, y \in X$,*

$$|f(x) - f(y)| < c|x - y|.$$

*Show that if $f$ is a contraction map and $x \in \mathbb{R}$, then the sequence*

$$x, f(x), f(f(x)), f(f(f(x))), \ldots$$

*is Cauchy.*

**Problem 4.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and a contraction map. Show that there exists $x \in \mathbb{R}$ such that $f(x) = x$. Why do you need continuity here?*

**Problem 5.** *Suppose*

$$\lim_{x \to p} f(x) = L \quad \text{and} \quad \lim_{x \to p} g(x) = M.$$

*Show that*

$$\begin{aligned} &\lim_{x \to p}(f(x) + g(x)) &&= L + M, \\ &\lim_{x \to p} f(x)g(x) &&= LM, \\ \text{and} \quad &\lim_{x \to p} \frac{1}{g(x)} &&= \tfrac{1}{M} \quad \text{if } M \neq 0. \end{aligned}$$

**Problem 6.** *Suppose*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \neq \mathbb{Q}. \end{cases}$$

*Show that $\lim_{x \to a} f(x)$ does not exist, for any $a \in \mathbb{R}$.*

**Problem 7.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.*

**Problem 8.** *Show that $\lim_{x \to p} f(x) = L$ if and only if for any sequence $\{a_n\}$ in $\mathbb{R} \setminus \{p\}$ converging to $p$, $\{f(a_n)\}$ converges to $L$.*

**Problem 9.** *In the spirit of the $\varepsilon - \delta$ definitions of $\lim_{x \to a} f(x) = L$, give definitions for*

$$\lim_{x \to \infty} f(x) = L \text{ and } \lim_{x \to a} f(x) = \infty.$$