1. Problem Set 9

Problem 1. *Suppose* $f(x) \leq g(x) \leq h(x)$ *for all* $x \in \mathbb{R}$, *and*

$$\lim_{x \to p} f(x) = \lim_{x \to p} h(x) = L.$$

Show that $\lim_{x \to p} g(x) = L$.

Problem 2. *Suppose*

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{otherwise.} \end{cases}$$

Show that f *is continuous at* 0 *but not differentiable there. Convince yourself (without writing anything down) that zero is the only point at which* f *is continuous.*

Problem 3. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is continuous, and* $f(p) > 0$. *Use the epsilon-delta definition of continuity to show that there exists an open interval* (a, b) *containing* p *such that* $f(x) > 0$ *for all* $x \in (a, b)$. *(Optional: do the same thing but using the open sets definition of continuity. Which is easier?)*

Problem 4. *Show that if* $y \geq 0$ *then there exists a unique* $x \geq 0$ *such that* $x^2 = y$. *Define* $\sqrt{y} = x$. *Show that the square root function is continuous at* x *for every* $x > 0$.

Problem 5. *Show that the square root function is differentiable at* x *for every* $x > 0$, *and find the derivative.*

Problem 6. *A function* $f : A \to \mathbb{R}$ *is Lipschitz if there exists* $C > 0$ *such that for all* $x, y \in A$

$$|f(x) - f(y)| \leq C|x - y|.$$

Show that if $f : \mathbb{R} \to \mathbb{R}$ *is differentiable, and* f' *is continuous, then the restriction of* f *to* $[a, b]$ *is Lipschitz.*

Problem 7. *Suppose* $A \subset \mathbb{R}$ *is open and* $f : A \to \mathbb{R}$ *is Lipschitz. Show that* f *is continuous.*

Problem 8. *Suppose* g *is differentiable at* a *and* f *is differentiable at* $g(a)$. *Define*

$$\phi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{f'(g(a))} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{otherwise} \end{cases}$$

Show that ϕ *is continuous at* 0.

Problem 9. *Use Problem 8 to prove the chain rule.*