1. Reals: Order and Limit Points

Definition 1.1 (Provisional Definition of $\mathbb{R}$)**.** *The* real numbers *are a nonempty set* $\mathbb{R}$ *together with a relation* $<$ *and two operations* $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *which satisfy the following axioms: ???*

Definition 1.2. *Let* X *be a set. An* ordering $<$ *on the set* X *is a relation on* X, *satisfying the following properties:*

(a) *For all* $x, y \in X$ *such that* $x \neq y$, *either* $x < y$ *or* $y < x$.
(b) *For all* $x, y \in X$, *if* $x < y$ *then* $x \neq y$.
(c) *For all* $x, y, z \in X$, *if* $x < y$ *and* $y < z$ *then* $x < z$.

Axiom 1: $<$ is an ordering on $\mathbb{R}$.

Proposition 1.1. *If* x *and* y *are points of* $\mathbb{R}$, *then* $x < y$ *and* $y < x$ *are not both true.*

Definition 1.3. *If* $A \subset \mathbb{R}$, *then a point* $a \in A$ *is a* first *point of* A *if, for every element* $x \in A$, *either* $a < x$ *or* $a = x$. *Similarly, a point* $b \in A$ *is called a* last *point of* A *if, for every* $x \in A$, *either* $x < b$ *or* $x = b$.

Lemma 1.2. *If* A *is a nonempty, finite subset of* $\mathbb{R}$, *then* A *has a first and last point.*

Proposition 1.3. *Suppose that* A *is a set of* n *distinct points in* $\mathbb{R}$. *Then symbols* $a_1, \ldots, a_n$ *may be assigned to each point of* A *so that* $a_1 < a_2 < \cdots < a_n$, *i.e.* $a_i < a_{i+1}$ *for* $1 \leq i \leq n-1$.

Definition 1.4. *If* $x, y, z \in \mathbb{R}$ *and both* $x < y$ *and* $y < z$, *then we say that* y *is* between x *and* z.

Corollary 1.4. *Of three distinct points, one must be between the other two.*

Axiom 2: $\mathbb{R}$ has no first or last point.

Definition 1.5. *If* $a, b \in \mathbb{R}$ *and* $a < b$, *then the set of points between* a *and* b *is called an* open interval, *denoted by* (a, b). *The set* $\{a\} \cup (a, b) \cup \{b\}$ *is called a* closed interval, *denoted by* $[a, b]$.

Proposition 1.5. *If* x *is a point of* $\mathbb{R}$, *then there exists an open interval* (a, b) *such that* $x \in (a, b)$.

Definition 1.6. *Let* A *be a nonempty subset of* $\mathbb{R}$. *A point* p *of* $\mathbb{R}$ *is called a* limit point *of* A *if every open interval* I *containing* p *has nonempty intersection with* $A \setminus \{p\}$. *Explicitly, this means:*

$$\text{for every open interval } I \text{ with } p \in I, \text{ we have } I \cap (A \setminus \{p\}) \neq \varnothing.$$

Notice that we do not require that a limit point p of A be an element of A.

Remark: Note that p is not a limit point of A if there exists an open interval (a, b) such that $p \in (a, b)$ and $(a, b) \cap A \setminus \{p\} = \varnothing$.

Proposition 1.6. *If* p *is a limit point of* A *and* $A \subset B$, *then* p *is a limit point of* B.

Lemma 1.7. *Suppose* (a, b) *is an open interval. Define the exterior of* (a, b) *to be the set* $\mathbb{R} \setminus [a, b]$*. Then no point in the exterior of* (a, b) *is a limit point of* (a, b)*, and no point of* (a, b) *is a limit point of the exterior of* (a, b)*.*

Proposition 1.8. *If two open intervals have a point* x *in common, their intersection is an open interval containing* x*.*

Corollary 1.9. *If* n *open intervals have a point* x *in common, their intersection is an open interval containing* x*.*

Theorem 1.10. *Let* $A, B \subset \mathbb{R}$*. If* p *is a limit point of* $A \cup B$*, then* p *is a limit point of* A *or* B*.*

Corollary 1.11. *Let* $A_1, \ldots, A_n$ *be* n *subsets of* $\mathbb{R}$*. Then* p *is a limit point of* $A_1 \cup \cdots \cup A_n$ *if and only if* p *is a limit point of at least one of the sets* A_k*.*

Proposition 1.12. *If* p *and* q *are distinct points of* $\mathbb{R}$*, then there exist disjoint open intervals* I_1 *and* I_2 *containing* p *and* q*, respectively.*

Corollary 1.13. *A subset of* $\mathbb{R}$ *consisting of one point has no limit points.*

Corollary 1.14. *A finite subset* $A \subset \mathbb{R}$ *has no limit points.*

Corollary 1.15. *If* $A \subset \mathbb{R}$ *is finite and* $x \in A$*, then there exists an open interval* I *such that* $A \cap I = \{x\}$*.*

Proposition 1.16. *If* p *is a limit point of* A *and* I *is an open interval containing* p*, then the set* $I \cap A$ *is infinite.*

2. Reals: Open and Closed

Definition 2.1. *A subset of* $\mathbb{R}$ *is* closed *if it contains all of its limit points.*

Theorem 2.1. *The sets* $\varnothing$ *and* $\mathbb{R}$ *are closed. Moreover a subset of* $\mathbb{R}$ *containing a finite number of points is closed.*

Definition 2.2. *Let* X *be a subset of* $\mathbb{R}$*. The* closure *of* X *is the subset* $\overline{X}$ *of* $\mathbb{R}$ *defined by:*

$$\overline{X} = X \cup \{x \in \mathbb{R} \mid x \text{ is a limit point of } X\}.$$

Proposition 2.2. $X \subset \mathbb{R}$ *is closed if and only if* $X = \overline{X}$*.*

Proposition 2.3. *The closure of* $X \subset \mathbb{R}$ *satisfies* $\overline{X} = \overline{\overline{X}}$*.*

Corollary 2.4. *Given any subset* $X \subset \mathbb{R}$*, the closure* $\overline{X}$ *is closed.*

Definition 2.3. *A subset* U *of* $\mathbb{R}$ *is* open *if its complement* $\mathbb{R} \setminus U$ *is closed.*

Theorem 2.5. *Let* $U \subset \mathbb{R}$*. Then* U *is open if and only if for all* $x \in U$*, there exists an open interval* I *such that* $x \in I \subset U$*.*

Corollary 2.6. *Every open interval is open. Every complement of an open interval is closed. Moreover* $\varnothing$ *and* $\mathbb{R}$ *are open.*

Theorem 2.7. *Let U be a nonempty open set. Then U is the union of a collection of open intervals.*

Theorem 2.8. *Let $\{X_\lambda\}$ be an arbitrary collection of closed subsets of $\mathbb{R}$. Then the intersection $\bigcap_\lambda X_\lambda$ is closed.*

Theorem 2.9. *Let $U_1, \ldots, U_n$ be a finite collection of open subsets $\mathbb{R}$. Then the intersection $U_1 \cap \cdots \cap U_n$ is open.*

Corollary 2.10. *Let $\{U_\lambda\}$ be an arbitrary collection of open subsets of $\mathbb{R}$. Then the union $\bigcup_\lambda U_\lambda$ is open. Let $X_1, \ldots, X_n$ be a finite collection of closed subsets of $\mathbb{R}$. Then the union $X_1 \cup \cdots \cup X_n$ is closed.*

Definition 2.4. *Let X be any set. A* topology *on X is a collection $\mathcal{T}$ of subsets of X that satisfy the following properties:*

1. *X and $\varnothing$ are elements of $\mathcal{T}$.*
2. *The union of an arbitrary collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.*
3. *The intersection of a finite number of sets in $\mathcal{T}$ is also in $\mathcal{T}$.*

The elements of $\mathcal{T}$ are called the open sets *of X. The set X with the structure of the topology $\mathcal{T}$ is called a* topological space[1].

[1]The word *topology* comes from the Greek word *topos* ($\tau \acute{o} \pi o \zeta$), which means "place".

3. Connectedness

Definition 3.1. *A set $X \subset \mathbb{R}$ is disconnected if it can be written in the form*

$$X \subset A \cup B$$

where A and B are open and disjoint, and $A \cap X$, $B \cap X$ are nonempty. X is connected if it is not disconnected.

Axiom 3: $\mathbb{R}$ is connected.

Proposition 3.1. *The only subsets of $\mathbb{R}$ that are both open and closed are $\varnothing$ and $\mathbb{R}$.*

Theorem 3.2. *For all $x, y \in \mathbb{R}$, if $x < y$, then there exists $z \in \mathbb{R}$ such that z is in between x and y.*

Corollary 3.3. *Every open interval is infinite.*

Corollary 3.4. *Every point of $\mathbb{R}$ is a limit point of $\mathbb{R}$.*

Corollary 3.5. *Every point of (a, b) is a limit point of (a, b).*

Definition 3.2. *Let X be a subset of $\mathbb{R}$. A point u is called an* upper bound *of X if for all $x \in X$, $x \leq u$. A point l is called a* lower bound *of X if for all $x \in X$, $l \leq x$. If there exists an upper bound of X, then we say that X is* bounded above. *If there exists a lower bound of X, then we say that X is* bounded below. *If X is bounded above and below, then we simply say that X is* bounded.

Definition 3.3. *Let X be a subset of $\mathbb{R}$. We say that u is the* least upper bound *of X and write $u = \sup X$ if:*

1. *u is an upper bound of X, and*
2. *if u' is an upper bound of X, then $u \leq u'$.*

We say that l is the greatest lower bound *and write $l = \inf X$ if:*

1. *l is a lower bound of X, and*
2. *if l' is a lower bound of X, then $l' \leq l$.*

Lemma 3.6. *Let $X \subset \mathbb{R}$ and define:*

$$\Psi(X) = \{x \in \mathbb{R} \mid x \text{ is not an upper bound of } X\}.$$

Then $\Psi(X)$ is open. Define:

$$\Omega(X) = \{x \in \mathbb{R} \mid x \text{ is not a lower bound of } X\}.$$

Then $\Omega(X)$ is open.

Theorem 3.7 (*). *Suppose that X is nonempty and bounded. Then $\sup X$ and $\inf X$ both exist.*

Theorem 3.8. *Let X be a subset of $\mathbb{R}$. Suppose that $\sup X$ exists and $\sup X \notin X$. Then $\sup X$ is a limit point of X. The same holds for $\inf X$.*

Corollary 3.9. *Both a and b are limit points of (a, b).*

Corollary 3.10. *Every nonempty closed and bounded set has a first point and a last point.*

4. Continuity

Definition 4.1. *If $f : A \to B$, and $X \subset B$ then the preimage of X is the set*

$$f^{-1}(X) = \{a \in A | f(a) \in X\}.$$

Exercise 4.1. *What is the relationship between $f(f^{-1}(X))$ and X? What is the relationship between $f^{-1}(f(X))$ and X?*

Lemma 4.2. *Suppose $f : A \to B$, and $X, Y \subset B$. Then*

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y) \text{ and } f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$$

Definition 4.2. *A function $f : \mathbb{R} \to \mathbb{R}$ is* continuous *if for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open.*

Theorem 4.3. *$f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all $x \in \mathbb{R}$ and every open interval I_1 containing $f(x)$, there exists an open interval I_2 containing x such that $f(I_2) \subset I_1$.*

Lemma 4.4. *Suppose $f : A \to B$, and $X, Y \subset A$. Then*

$$f(X \cap Y) \subset f(X) \cap f(Y).$$

Theorem 4.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that x is a limit point of $A \subset \mathbb{R}$. Then $f(x)$ is a limit point of $f(A)$ or $f(x) \in f(A)$.*

Theorem 4.6. *Every open interval (a, b) is connected.*

Theorem 4.7. *Suppose that $X \subset \mathbb{R}$ is a connected subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then $f(X)$ is connected.*

Corollary 4.8 (Intermediate Value Theorem)**.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $a, b \in \mathbb{R}$ such that $a < b$. Then if y is between $f(a)$ and $f(b)$ then there exists $c \in (a, b)$ such that $f(c) = y$.*

5. Compactness

Definition 5.1. *Let $X \subset \mathbb{R}$, and suppose $\mathcal{O} = \{U_\lambda\}$ is a collection of open sets in $\mathbb{R}$. We say $\mathcal{O}$ is an open cover of $\mathbb{R}$ if*

$$X \subset \bigcup_\lambda U_\lambda.$$

Definition 5.2. *Let X be a subset of $\mathbb{R}$. X is* compact *if for every open cover $\mathcal{O}$ of X, there exists a finite subset $\mathcal{O}' \subset \mathcal{O}$ that is also an open cover.*

Proposition 5.1. $\mathbb{R}$ *is not compact.*

Theorem 5.2. *If X is compact, then X is bounded.*

Recall that $\operatorname{ext}(a, b)$ refers to the set $\mathbb{R} \setminus [a, b]$, and by a homework problem from problem set 2, this is an open set.

Lemma 5.3. *Let $p \in \mathbb{R}$ and consider the set:*

$$\mathcal{O} = \{\operatorname{ext}(a, b) \mid p \in (a, b)\}.$$

No finite subset of $\mathcal{O}$ covers $\mathbb{R} \setminus \{p\}$.

Proposition 5.4. *No open interval (a, b) is compact.*

Theorem 5.5. *If X is compact, then X is closed.*

Proposition 5.6. *The set $[a, b]$ is compact.*

Theorem 5.7 (Heine-Borel)**.** *Let $X \subset \mathbb{R}$. X is compact if and only if X is closed and bounded.*

Theorem 5.8 (*)**.** *Suppose $X \subset \mathbb{R}$ is compact, and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then $f(X)$ is compact.*

Corollary 5.9 (Extreme Value Theorem*)**.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $[a, b]$ is a closed interval. Show that $f[a, b]$ has a first point and a last point.*

Theorem 5.10 (Bolzano-Weierstrass)**.** *Every bounded infinite subset of $\mathbb{R}$ has at least one limit point.*

6. Field Axioms

Definition 6.1. *Suppose F is a nonempty set with two binary operations $+$ and $\cdot$. We say that F is a field if it satisfies the following 10 axioms:*

Field Axiom 1: (Commutativity of Addition) For all $x, y \in \mathbb{R}$, we have $x+y = y+x$.

Field Axiom 2: (Associativity of Addition) For all $x, y, z \in \mathbb{R}$, we have $(x+y)+z = x+(y+z)$.

Field Axiom 3: (Additive Identity) There exists $0 \in \mathbb{R}$ such that $0 + x = x$ for all $x \in \mathbb{R}$.

Field Axiom 4: (Additive Inverses) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x + y = 0$. In this case we define $-x := y$.

Field Axiom 5: (Commutativity of Multiplication) For all $x, y \in \mathbb{R}$, we have $x \cdot y = y \cdot x$.

Field Axiom 6: (Associativity of Multiplication) For all $x, y, z \in \mathbb{R}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Field Axiom 7: (Multiplicative Identity) There exists $1 \in \mathbb{R}$ such that $1 \cdot x = x$ for all $x \in \mathbb{R}$.

Field Axiom 8: (Multiplicative Inverses) For all $x \in \mathbb{R}$ such that $x \neq 0$, there exists $y \in \mathbb{R}$ such that $x \cdot y = 1$. In this case we define $x^{-1} := y$.

Field Axiom 9: (Distributivity) For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Field Axiom 10: (Distinctiveness of Identity) $1 \neq 0$.

The fourth axiom of the real numbers is that the real numbers form a field:

Axiom 4: The real numbers, with the binary operations $+$ and $\cdot$, form a field.

We are almost done with axioms for the real numbers. We just need to specify one more thing – that the field operations interact nicely with order.

Definition 6.2. *We say that a field F, together with a relation $<$, is an ordered field, if $<$ is an ordering on F, and*

- *Addition respects the ordering: if $x < y$, then $x + z < y + z$ for all $z \in F$.*
- *Multiplication respects the ordering: if $0 < x$ and $0 < y$, then $0 < x \cdot y$.*

Axiom 5: $\mathbb{R}$, with the order $<$, and the binary operations $+$ and $\cdot$, is an ordered field.

Note that $\mathbb{R}$ contains a natural copy of $\mathbb{N}$:

Proposition 6.1. *Define*

$$i : \mathbb{N} \to \mathbb{R}$$

by $i(1) = 1$ *and* $i(n) = i(n-1) + 1$*. Then* i *is injective and* $i(n+m) = i(n) + i(m)$*.*

The image of $\mathbb{N}$ under this map acts exactly like $\mathbb{N}$: we will refer to it as $\mathbb{N}$ as well. This is arguably terrible notation but in practice it will turn out that this is not confusing.

Proposition 6.2. *For all* $x \in \mathbb{R}$*, there exists* $n \in \mathbb{N}$ *such that* $n > x$*.*

We can define $\mathbb{Z}$ and $\mathbb{Q}$ inside $\mathbb{R}$ as well:

Definition 6.3. *We define*

$$\mathbb{Z} = \{x \in \mathbb{R} | x \in \mathbb{N} \text{ or } -x \in \mathbb{N}\}$$

and

$$\mathbb{Q} = \{pq^{-1} \in \mathbb{R} | p, q \in \mathbb{Z}, q \neq 0\}.$$

Definition 6.4. *We say that* A *is dense in* $\mathbb{R}$ *if every open interval in* $\mathbb{R}$ *contains an element of* A*.*

Theorem 6.3. $\mathbb{Q}$ *is dense in* $\mathbb{R}$*.*

Adapted from notes by John Boller, Daniele Rosso, John Lind, and Francis Chung.