1. Sequences

Definition 1.1. *A* sequence *is a function* $a : N \to \mathbb{R}$ *from the natural numbers to the real numbers.*

By setting $a_n = a(n)$, we think of a sequence a as a list $a_1, a_2, a_3 \ldots$ of real numbers. We use the notation $\{a_n\}_{n=1}^{\infty}$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply $\{a_n\}$. More generally, we also use the term sequence to refer to a function defined on $\{n \in N | n \geq n_0\}$ for any fixed $n_0 \in \mathbb{N}$. We denote this by writing $\{a_n\}_{n=n_0}^{\infty}$ for such a sequence.

Definition 1.2. *We say that a sequence* $\{a_n\}$ converges *to a point* $p \in \mathbb{R}$ *if for every open interval* I *containing* p*, there exists* $N \in \mathbb{N}$ *such that if* $n \geq N$*, then* $a_n \in I$*.*

If $\{a_n\}$ *converges to* p*, we write this as:*

$$\lim_{n\to\infty} a_n = p,$$

and call p *the* limit *of* $\{a_n\}$*. If* $\{a_n\}$ *does not converge to any point* p*, we call it* divergent*.*

Remark: Note that if a sequence $\{a_n\}$ converges to p, then any region containing p contains all but finitely many terms in the sequence.

Theorem 1.1. *Suppose that*

$$\lim_{n\to\infty} a_n = p \quad \text{and} \quad \lim_{n\to\infty} a_n = p'.$$

Then $p = p'$*. In other words, limits of sequences are unique.*

Theorem 1.2 (*)**.** *Suppose* $\lim\limits_{n\to\infty} a_n = p$ *and* $f : \mathbb{R} \to \mathbb{R}$ *is continuous. Then* $\{f(a_n)\}$ *converges to* $f(p)$*.*

Sometimes divergent sequences have points that behave like limits, but are not necessarily unique:

Definition 1.3. *A point* $p \in \mathbb{R}$ *is an* accumulation point *of* $\{a_n\}$ *if for every open interval* I *containing* p*, there exists infinitely many* $n \in \mathbb{N}$ *with* $a_n \in I$*.*

Exercise 1.3. *Construct a sequence with two distinct accumulation points. Construct a sequence with infinitely many accumulation points. Construct a sequence with no accumulation points.*

Theorem 1.4. *Suppose that* $\lim_{n\to\infty} a_n = p$*. Then* p *is the only accumulation point of the sequence* $\{a_n\}$*.*

Definition 1.4. *Let* (a_n) *be a sequence. A* subsequence *of* $\{a_n\}$ *is a sequence* b *defined by the composition* $b = a \circ i \colon \mathbb{N} \to \mathbb{R}$*, where* $i \colon \mathbb{N} \to \mathbb{N}$ *is an increasing function. (By increasing, we mean that* i *has the property that if* $n < m$*, then* $i(n) < i(m)$*.)*

If we let $n_k = i(k) \in \mathbb{N}$, we can write $b_k = a_{n_k}$, so that $\{b_n\}$ is the sequence $b_1, b_2, b_3, \ldots,$ which is equal to the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$ where $n_1 < n_2 < n_3 < \cdots$.

Exercise 1.5. *Construct a divergent sequence with a subsequence which converges.*

Theorem 1.6. *If* $\{a_n\}$ *converges to* p*, then so does any of its subsequences.*

Theorem 1.7. *Let* $\{a_n\}$ *be a sequence and suppose that there exists a subsequence* ($b_k = a_{n_k}$) *that converges to* p*. Then* p *is an accumulation point of* (a_n)*.*

Lemma 1.8. *Let* $p \in [a, b]$ *and define* $I_k^p = (p - 1/k, p + 1/k)$*. Then* $\cap_{k\in\mathbb{N}} I_k^p = \{p\}$*, and moreover for any* (a, b) *containing* p*, there exists* $K \in \mathbb{N}$ *such that for all* $k > K$*,* $I_k^p \subset (a, b)$*.*

Theorem 1.9. p *is an accumulation point of* $\{a_n\}$ *if and only if there exists a subsequence* b_k *converging to* p*.*

Definition 1.5. *A sequence* $\{a_n\}$ *is bounded if the set of all* a_n *is bounded. Similar definitions apply for bounded above and bounded below.*

Proposition 1.10. *Suppose* $\{a_n\}$ *is nondecreasing (meaning that* $a_n \leq a_{n+1}$ *for each* n*) and bounded above. Then* $\{a_n\}$ *converges.*

Theorem 1.11. *Suppose* $\{a_n\}$ *converges. Then* $\{a_n\}$ *is bounded.*

Theorem 1.12 (Bolzano-Weierstrass*)**.** *Every bounded sequence has a convergent subsequence.*

2. Metric (Re)Definitions and Cauchy sequences

Theorem 2.1. *A sequence $\{a_n\}$ converges to p if and only if for all $\varepsilon > 0$ there exists N such that for $n > N$,*

$$|a_n - p| < \varepsilon.$$

Remark: One possible exercise is to prove the theorems of the previous section using this definition.

Definition 2.1. *A sequence $\{a_n\}$ is Cauchy if for all $\varepsilon > 0$ there exists N such that for all $n, m > N$,*

$$|a_n - a_m| < \varepsilon.$$

Lemma 2.2. *Suppose $\{a_n\}$ is Cauchy and a subsequence of $\{a_n\}$ converges to p. Then $\{a_n\}$ converges to p.*

Lemma 2.3. *If $\{a_n\}$ is Cauchy then $\{a_n\}$ is bounded.*

Theorem 2.4 (*). *A sequence $\{a_n\}$ is Cauchy if and only if it converges.*

3. Series

Definition 3.1. *Consider a sequence* $\{a_n\}$*. We define the* n^{th} *partial sum of* $\{a_n\}$ *by*

$$s_n = a_1 + \ldots + a_n.$$

We say that $\{a_n\}$ *is summable (or* $\sum_{n=1}^{\infty} a_n$ *converges) if* $\{s_n\}$ *converges, and then we define*

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Exercise 3.1. *Convince yourself that this definition is sensible. Prove that if* $\{a_n\}$ *and* $\{b_n\}$ *are summable then so is* $\{a_n + b_n\}$*, and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Proposition 3.2. *If* $\{a_n\}$ *is summable then* $\lim_{n\to\infty} a_n = 0$*. Note that the converse is false (why?).*

Proposition 3.3. *Suppose* $\{a_n\}$ *is nonnegative and the sequence of its partial sums* $\{s_n\}$ *is bounded. Then* $\{a_n\}$ *is summable.*

Theorem 3.4 (Comparison Test)**.** *Suppose* $0 \le a_n \le b_n$ *for all* $n \in \mathbb{N}$*, and* $\{b_n\}$ *is summable. Then* $\{a_n\}$ *is summable.*

Theorem 3.5. *Suppose* $0 \le a_n, b_n$ *for all* $n \in \mathbb{N}$*, and* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and is non zero. Then* $\{a_n\}$ *is summable if and only if* $\{b_n\}$ *is summable.*

Lemma 3.6. *The sum* $\sum_{n=1}^{\infty} r^n$ *converges if* $0 \le r < 1$ *and diverges if* $r \ge 1$*.*

Theorem 3.7 (Ratio Test*)**.** *Suppose* $0 \le a_n$ *for all* $n \in \mathbb{N}$*, and* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$*. Then* $\{a_n\}$ *is summable if* $r < 1$ *and not summable if* $r > 1$*. If* $r = 1$ *then* $\{a_n\}$ *may or may not be summable.*

Theorem 3.8 (Integral Test)**.** *Suppose* f *is positive, integrable, and decreasing on* $[1, x]$*, for any* $x > 1$*, and* $a_n = f(n)$*. Then* $\{a_n\}$ *is summable if and only if*

$$\lim_{x\to\infty} \int_1^x f(t)dt$$

exists.

Theorem 3.9. *Suppose* $\sum_{n=1}^{\infty} |a_n|$ *converges. Then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Remark: If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely. It is possible for a series to converge but not absolutely – the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \dots$ is a good example. However, it is possible to show that such a series can be rearranged to converge to any number you want – which is terribly depraved behaviour. Mostly we would like our series to converge absolutely if we can arrange it.

4. Sequences of Functions

Definition 4.1. *Suppose $f_n, f : A \to \mathbb{R}$. We say that f_n converge to f pointwise on A if*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in A$.

Definition 4.2. *Suppose $f_n, f : A \to \mathbb{R}$. We say that f_n converge to f uniformly on A if for every $\varepsilon > 0$ there exists N such that for all $n > N$ and $x \in A$,*

$$|f_n(x) - f(x)| < \varepsilon.$$

Exercise 4.1. *Give an example of a sequence of functions that converges pointwise on $[a, b]$ but not uniformly.*

Theorem 4.2 (*)**.** *Suppose $f_n : A \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly on A. Then f is continuous on A.*

Theorem 4.3. *Suppose $f_n, f : [a, b] \to \mathbb{R}$ are integrable and $f_n \to f$ uniformly on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Theorem 4.4. *Suppose $f_n, f : (a, b) \to \mathbb{R}$, $f_n \to f$ pointwise, and each f_n is differentiable, and $f_n' \to f$ uniformly on (a, b). Then f is differentiable on $[a, b]$, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

for all $x \in (a, b)$.