1. Ordering

Definition 1.1. *A continuum is a nonempty set $\mathcal{C}$ together with a relation $<$, which satisfies the following axioms:*

OA1: *For all $x, y \in \mathcal{C}$ such that $x \neq y$, either $x < y$ or $y < x$.*
OA2: *For all $x, y \in \mathcal{C}$, if $x < y$ then $x \neq y$.*
OA3: *For all $x, y, z \in \mathcal{C}$, if $x < y$ and $y < z$ then $x < z$.*
IA: *$\mathcal{C}$ has no first or last points. (See Definition 1.2)*
CA: *$\mathcal{C}$ is connected. (See Definition 4.1)*

A relation which satisfies axioms OA1-3 is called an *ordering*. Technically the "or" which appears in OA1 is an inclusive or, but the next proposition shows that it is secretly the exclusive or.

Proposition 1.1. *If x and y are points of $\mathcal{C}$, then $x < y$ and $y < x$ are not both true.*

The next definition explains the meaning of IA.

Definition 1.2. *If $A \subseteq \mathcal{C}$, then a point $a \in A$ is a* first *point of A if, for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a* last *point of A if, for every $x \in A$, either $x < b$ or $x = b$.*

Lemma 1.2. *If A is a nonempty, finite subset of $\mathcal{C}$, then A has a first and last point.*

Proposition 1.3. *Suppose that A is a set of n distinct points in $\mathcal{C}$. Then symbols $a_1, \ldots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e. $a_i < a_{i+1}$ for $1 \leq i \leq n-1$.*

Definition 1.3. *If $x, y, z \in \mathcal{C}$ and both $x < y$ and $y < z$, then we say that y is* between *x and z.*

Corollary 1.4. *Of three distinct points, one must be between the other two.*

Definition 1.4. *If $a, b \in \mathcal{C}$ and $a < b$, then the open interval (a, b) is defined by*

$$(a, b) = \{x \in \mathcal{C} | a < x < b\}.$$

The closed interval $[a, b]$ is defined by

$$[a, b] = \{x \in \mathcal{C} | a < x < b\}.$$

Proposition 1.5. *If x is a point of $\mathcal{C}$, then there exists an open interval (a, b) such that $x \in (a, b)$.*

2. Limit Points

Definition 2.1. *Let A be a nonempty subset of $\mathcal{C}$. A point p of $\mathcal{C}$ is called a* limit point *of A if every open interval I containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:*

$$\text{for every open interval } I \text{ with } p \in I, \text{ we have } I \cap (A \setminus \{p\}) \neq \varnothing.$$

Notice that we do not require that a limit point p of A be an element of A.

Remark: Note that p is not a limit point of A if there exists an open interval (a, b) such that $p \in (a, b)$ and $(a, b) \cap A \setminus \{p\} = \varnothing$.

Proposition 2.1. *If p is a limit point of A and $A \subset B$, then p is a limit point of B.*

Lemma 2.2. *Suppose (a, b) is an open interval. Define the exterior of (a, b) to be the set $\mathcal{C} \setminus [a, b]$. Then no point in the exterior of (a, b) is a limit point of (a, b), and no point of (a, b) is a limit point of the exterior of (a, b).*

Proposition 2.3. *If two open intervals have a point x in common, their intersection is an open interval containing x.*

Corollary 2.4. *If n open intervals have a point x in common, their intersection is an open interval containing x.*

Theorem 2.5 (*)**.** *Let $A, B \subset \mathcal{C}$. If p is a limit point of $A \cup B$, then p is a limit point of A or B.*

Corollary 2.6. *Let $A_1, \ldots, A_n$ be n subsets of $\mathcal{C}$. Then p is a limit point of $A_1 \cup \cdots \cup A_n$ if and only if p is a limit point of at least one of the sets A_k.*

Proposition 2.7. *If p and q are distinct points of $\mathcal{C}$, then there exist disjoint open intervals I_1 and I_2 containing p and q, respectively.*

Corollary 2.8. *A subset of $\mathcal{C}$ consisting of one point has no limit points.*

Corollary 2.9. *A finite subset $A \subset \mathcal{C}$ has no limit points.*

Corollary 2.10. *If $A \subset \mathcal{C}$ is finite and $x \in A$, then there exists an open interval I such that $A \cap I = \{x\}$.*

Proposition 2.11. *If p is a limit point of A and I is an open interval containing p, then the set $I \cap A$ is infinite.*

3. Topology

Definition 3.1. *A subset of $\mathcal{C}$ is* closed *if it contains all of its limit points.*

Theorem 3.1. *The sets $\varnothing$ and $\mathcal{C}$ are closed. Moreover a subset of $\mathcal{C}$ containing a finite number of points is closed.*

Definition 3.2. *Let X be a subset of $\mathcal{C}$. The* closure *of X is the subset $\overline{X}$ of $\mathcal{C}$ defined by:*

$$\overline{X} = X \cup \{x \in \mathcal{C} \mid x \text{ is a limit point of } X\}.$$

Proposition 3.2. *$X \subset \mathcal{C}$ is closed if and only if $X = \overline{X}$.*

Proposition 3.3. *The closure of $X \subset \mathcal{C}$ satisfies $\overline{X} = \overline{\overline{X}}$.*

Corollary 3.4. *Given any subset $X \subset \mathcal{C}$, the closure $\overline{X}$ is closed.*

Definition 3.3. *A subset U of $\mathcal{C}$ is* open *if its complement $\mathcal{C} \setminus U$ is closed.*

Theorem 3.5 (*). *Let $U \subset \mathcal{C}$. Then U is open if and only if for all $x \in U$, there exists an open interval I such that $x \in I \subset U$.*

Corollary 3.6. *Every open interval is open. Every complement of an open interval is closed. Moreover $\varnothing$ and $\mathcal{C}$ are open.*

Theorem 3.7. *Let $\{U_\lambda\}$ be an arbitrary collection of open subsets of $\mathcal{C}$. Then the union $\bigcup_\lambda U_\lambda$ is open.*

Corollary 3.8. *Let $\{X_\lambda\}$ be an arbitrary collection of closed subsets of $\mathcal{C}$. Then the intersection $\bigcap_\lambda X_\lambda$ is closed.*

Theorem 3.9. *Let U be a nonempty open set. Then U is the union of a collection of open intervals.*

Theorem 3.10. *Let $U_1, \ldots, U_n$ be a finite collection of open subsets $\mathcal{C}$. Then the intersection $U_1 \cap \cdots \cap U_n$ is open.*

Corollary 3.11. *Let $X_1, \ldots, X_n$ be a finite collection of closed subsets of $\mathcal{C}$. Then the union $X_1 \cup \cdots \cup X_n$ is closed.*

Definition 3.4. *Let X be any set. A* topology *on X is a collection $\mathcal{T}$ of subsets of X that satisfy the following properties:*

(1) *X and $\varnothing$ are elements of $\mathcal{T}$.*
(2) *The union of an arbitrary collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.*
(3) *The intersection of a finite number of sets in $\mathcal{T}$ is also in $\mathcal{T}$.*

The elements of $\mathcal{T}$ are called the open sets *of X. The set X with the structure of the topology $\mathcal{T}$ is called a* topological space[1].

4. Connectedness

Definition 4.1. *Suppose $X \subseteq \mathcal{C}$. We say X is disconnected if there exist open sets $A, B \subset \mathcal{C}$ such that*

$$\begin{aligned} X &\subseteq A \cup B \\ A \cap B &= \varnothing \\ A \cap X, B \cap X &\neq \varnothing. \end{aligned}$$

We say X is connected if it is not disconnected.

Proposition 4.1. *The only subsets of $\mathcal{C}$ that are both open and closed are $\varnothing$ and $\mathcal{C}$.*

Theorem 4.2. *For all $x, y \in \mathcal{C}$, if $x < y$, then there exists $z \in \mathcal{C}$ such that z is in between x and y.*

Corollary 4.3. *Every open interval is infinite.*

Corollary 4.4. *Every point of $\mathcal{C}$ is a limit point of $\mathcal{C}$.*

Corollary 4.5. *Every point of (a, b) is a limit point of (a, b).*

[1]The word *topology* comes from the Greek word *topos* ($\tau\acute{o}\pi o\varsigma$), which means “place”.

Definition 4.2. *Let X be a subset of $\mathcal{C}$. A point u is called an* upper bound *of X if for all $x \in X$, $x \leq u$. A point l is called a* lower bound *of X if for all $x \in X$, $l \leq x$. If there exists an upper bound of X, then we say that X is* bounded above*. If there exists a lower bound of X, then we say that X is* bounded below*. If X is bounded above and below, then we simply say that X is* bounded*.*

Definition 4.3. *Let X be a subset of $\mathcal{C}$. We say that u is the* least upper bound *of X and write $u = \sup X$ if:*

(1) *u is an upper bound of X, and*
(2) *if u' is an upper bound of X, then $u \leq u'$.*

We say that l is the greatest lower bound *and write $l = \inf X$ if:*

(1) *l is a lower bound of X, and*
(2) *if l' is a lower bound of X, then $l' \leq l$.*

Lemma 4.6 (*)**.** *Let $X \subset \mathcal{C}$ and define:*

$$\Psi(X) = \{x \in \mathcal{C} \mid x \text{ is not an upper bound of } X\}$$

and

$$\Omega(X) = \{x \in \mathcal{C} \mid x \text{ is not a lower bound of } X\}.$$

Then $\Psi(X)$ and $\Omega(X)$ are open.

Theorem 4.7 (*)**.** *Suppose that X is nonempty and bounded. Then $\sup X$ and $\inf X$ both exist.*

Theorem 4.8 (*)**.** *Let X be a subset of $\mathcal{C}$. Suppose that $\sup X$ exists and $\sup X \notin X$. Then $\sup X$ is a limit point of X. The same holds for $\inf X$.*

Corollary 4.9. *Both a and b are limit points of (a, b).*

Corollary 4.10. *Every nonempty closed and bounded set has a first point and a last point.*

Theorem 4.11. *Every closed interval $[a, b]$ is connected.*

5. Continuity

Definition 5.1. *If $f : A \to B$, and $X \subset B$ then the preimage of X is the set*

$$f^{-1}(X) = \{a \in A | f(a) \in X\}.$$

Lemma 5.1. *Suppose $f : A \to B$, and $X, Y \subset B$. Then*

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y) \text{ and } f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$$

Lemma 5.2. *Suppose $f : A \to B$, and $X, Y \subset A$. Then*

$$f(X \cap Y) \subseteq f(X) \cap f(Y).$$

Definition 5.2. *A function $f : \mathcal{C} \to \mathcal{C}$ is* continuous *if for every open set $U \subset \mathcal{C}$, the preimage $f^{-1}(U)$ is open.*

Theorem 5.3 (*)**.** *Suppose that $X \subseteq \mathcal{C}$ is a connected subset of $\mathcal{C}$ and $f : \mathcal{C} \to \mathcal{C}$ is continuous. Then $f(X)$ is connected.*

Corollary 5.4 (Intermediate Value Theorem)**.** *Suppose $f : \mathcal{C} \to \mathcal{C}$ is continuous, and $[a,b] \subset \mathcal{C}$ is a nonempty closed interval. Then if y is between $f(a)$ and $f(b)$ then there exists $c \in [a,b]$ such that $f(c) = y$.*

Theorem 5.5 (*)**.** *$f : \mathcal{C} \to \mathcal{C}$ is continuous if and only if for all $x \in \mathcal{C}$ and every open interval I_1 containing $f(x)$, there exists an open interval I_2 containing x such that $f(I_2) \subset I_1$.*

Theorem 5.6. *Let $f : \mathcal{C} \to \mathcal{C}$ be continuous and suppose that x is a limit point of $A \subseteq \mathcal{C}$. Then $f(x)$ is a limit point of $f(A)$ or $f(x) \in f(A)$.*

6. Compactness

Definition 6.1. *Let $X \subset \mathcal{C}$, and suppose $\mathcal{O} = \{U_\lambda\}$ is a collection subsets of $\mathcal{C}$. We say $\mathcal{O}$ is an open cover of $\mathbb{R}$ if i) every U_λ is open and ii)*

$$X \subset \bigcup_\lambda U_\lambda.$$

Definition 6.2. *Let X be a subset of $\mathcal{C}$. X is* compact *if for every open cover $\mathcal{O}$ of X, there exists a finite subset $\mathcal{O}' \subset \mathcal{O}$ that is also an open cover.*

Proposition 6.1. *Any finite subset of $\mathcal{C}$ is compact.*

Proposition 6.2. *$\mathcal{C}$ is not compact.*

Theorem 6.3 (*)**.** *If X is compact, then X is bounded.*

Lemma 6.4. *Let $p \in \mathcal{C}$ and consider the set:*

$$\mathcal{O} = \{\text{ext}\,(a,b) \mid p \in (a,b)\}.$$

No finite subset of $\mathcal{O}$ covers $\mathcal{C} \setminus \{p\}$.

Proposition 6.5. *No open interval (a,b) is compact.*

Theorem 6.6. *If X is compact, then X is closed.*

Proposition 6.7. *The set $[a,b]$ is compact.*

Theorem 6.8 (Heine-Borel)**.** *Let $X \subset \mathcal{C}$. X is compact if and only if X is closed and bounded.*

Theorem 6.9 (*)**.** *Suppose $X \subset \mathcal{C}$ is compact, and $f : \mathcal{C} \to \mathcal{C}$ is continuous. Then $f(X)$ is compact.*

Corollary 6.10 (Extreme Value Theorem)**.** *Suppose $f : \mathcal{C} \to \mathcal{C}$ is continuous, and $[a,b]$ is a closed interval. Then $f[a,b]$ has a maximum and a minimum.*

Theorem 6.11 (Bolzano-Weierstrass*)**.** *Every bounded infinite subset of $\mathcal{C}$ has at least one limit point.*