1. L'Hôpital's Rule

Theorem 1.1. *Suppose* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, *and*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ exists.}$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof. First note that we can redefine $f(a) = g(a) = 0$ without changing anything in the statement of the theorem, since none of the limits in the statement are affected by the value of f or g at a. Note also that if we make this redefinition then f and g are continuous at a.

Now set

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon. \tag{1.1}$$

Note that implicit in this statement is the fact that in the set

$$J = \{x | 0 < |x-a| < \delta\},$$

the quantity $\dfrac{f'(x)}{g'(x)}$ is defined. In particular, f and g are differentiable on J and $g'(x) \neq 0$ on J.

Ok. Now suppose that $0 < |y-a| < \delta$. Then either $y \in (a, a+\delta)$ or $y \in (a-\delta, a)$. Suppose $y \in (a, a+\delta)$; the proof for the other case is similar. Since $(a, y] \subset J$, it follows that f and g are differentiable on (a, y) and continuous on $(a, y]$. Moreover f and g are continuous at a, so f and g are continuous on $[a, y]$.

Therefore the Cauchy Mean Value Theorem applies to f and g on the interval $[a, y]$, and so there exists $c \in (a, y)$ such that

$$f'(c)(g(y) - g(a)) = g'(c)(f(y) - f(a)).$$

Since $g(a) = f(a) = 0$, we have

$$f'(c)g(y) = g'(c)f(y). \tag{1.2}$$

Now $c \in (a, y) \subset (a, a+\delta) \subset J$, so the commentary after (1.1) implies that $g'(c) \neq 0$. Moreover if $g(y) = 0$, then applying the Mean Value Theorem to g between a and y implies that $g'(x) = 0$ for some $x \in (a, y) \subset (a, a+\delta) \subset J$, which contradicts the commentary after (1.1). Therefore $g(y) \neq 0$. Therefore we can divide (1.2) by $g'(c)$ and $g(y)$, to get

$$\frac{f'(c)}{g'(c)} = \frac{f(y)}{g(y)}.$$

Then

$$\left|\frac{f(y)}{g(y)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon,$$

where the inequality on the right follows from (1.1) and the fact that $c \in (a, y) \subset (a, a+\delta)$.

Therefore

$$\text{if } 0 < |y - a| < \delta \text{ then } \left|\frac{f(y)}{g(y)} - L\right| < \epsilon,$$

which finishes the proof.

$\square$