MA 575 Midterm Exam.

November 10 2017

Name: ______________________________

Problem 1. *Suppose* $f(x) \leq g(x) \leq h(x)$ *for all* $x \in \mathbb{R}$ *and* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. *Show that* $\lim_{x\to a} g(x)$ *exists and is also equal to* L.

Proof: Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \varepsilon$. Similarly, since $\lim_{x\to a} h(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|h(x) - L| < \varepsilon$.

Set $\delta = \min\{\delta_1, \delta_2\}$. Now if $x \in \mathbb{R}$ such that $0 < |x-a| < \delta$, then $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$, so

$$|f(x) - L| < \varepsilon \quad \text{and } |h(x) - L| < \varepsilon.$$

In other words

$$L - \varepsilon < f(x) < L + \varepsilon \quad \text{and } L - \varepsilon < h(x) < L + \varepsilon.$$

Now $f(x) \leq g(x) \leq h(x)$, so it follows that

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon.$$

Therefore

$$|g(x) - L| < \varepsilon,$$

and so we showed that if $0 < |x-a| < \delta$ then $|g(x) - L| < \varepsilon$.

Problem 2. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has the property that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Show that f is differentiable at* 0.

Proof: Notice that in particular $|f(0)| \leq 0^2$, so $f(0) = 0$. Now

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h)}{h}. \end{aligned}$$

Since $|f(h)| \leq h^2$, it follows that

$$-h \leq \frac{f(h)}{h} \leq h \quad \text{for } h > 0$$

and

$$-h \leq \frac{f(h)}{h} \leq -h \quad \text{for } h < 0$$

Therefore for all $h \neq 0$

$$-|h| \leq \frac{f(h)}{h} \leq |h|$$

and it follows from the squeeze theorem that

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = 0.$$

Problem 3. *Suppose f is continuous on $[0, \infty)$ and*

$$\lim_{x \to \infty} f(x) = L$$

for some $L \in \mathbb{R}$. Show that f is bounded on $[0, \infty)$.

Proof: Since $\lim_{x \to \infty} f(x) = L$, there exists N such that if $x > N$ then $|f(x) - L| < 1$. In other words, if $x > N$ then $L - 1 < f(x) < L + 1$.

Now f is continuous on $[0, N]$, so by the extreme value theorem, there exists m and M such that

$$m \leq f(x) \leq M \qquad \text{for all } x \in [0, N].$$

Now take $q = \min\{m, L - 1\}$ and $Q = \max\{M, L + 1\}$. Then if $x \in [0, \infty)$ then either $x \in [0, N]$ and $q \leq m \leq f(x) \leq M \leq Q$ or $x \in (N, \infty)$ and $q \leq L - 1 \leq f(x) \leq L + 1 \leq Q$. Therefore f is bounded on $[0, \infty)$ below by q and above by Q.

Problem 4. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is continuous everywhere and for all* $a, b \in \mathbb{R}$,

$$\int_a^b f(t)dt = 0.$$

Show that $f(t) = 0$ *for all* t.

Proof 1: Suppose there exists t_0 such that $f(t_0) > 0$. Then there exists $\delta > 0$ such that for all t such that $|t - t_0| < \delta$, we have $|f(t) - f(t_0)| > f(t_0)/2$. Therefore for all $t \in (t_0 - \delta, t_0 + \delta)$, we have $f(t) > f(t_0)/2$.

Therefore

$$\int_{t_0-\delta}^{t_0+\delta} f(t)dt \geq \delta f(t_0) > 0$$

and this is a contradiction.

Proof 2: Pick $a \in \mathbb{R}$ and define $F(x) = \displaystyle\int_a^x f(t)dt$. By the hypotheses on f, it follows that F is the zero function. Then since f is continuous, it follows from the 1st Fundamental Theorem of Calculus that $f(x) = F'(x) = 0$ for all x, so $f(x) = 0$ for all x.

Problem 5. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing on $[0,1]$; in other words if $x, y \in [0,1]$ with $x \leq y$ then $f(x) \leq f(y)$. Show that f is integrable on $[0,1]$.*

Proof: Note that since f is nondecreasing, f is bounded above by $f(1)$ and below by $f(0)$, so f is bounded. Now let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $n > (f(1) - f(0))/\varepsilon$, and consider the partition $P = \{0, 1/n, 2/n, \ldots, 1\}$.

Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) - \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\ &= \frac{1}{n}\sum_{i=1}^{n}(M_i - m_i). \end{aligned}$$

Since f is nondecreasing, $M_i = \sup\{f(x) | x \in [t_{i-1}, t_i]\} = f(t_i)$ and $m_i = \inf\{f(x) | x \in [t_{i-1}, t_i]\} = f(t_{i-1})$. Therefore

$$U(f,P) - L(f,P) = \frac{1}{n}\sum_{i=1}^{n}(f(t_i) - f(t_{i-1})).$$

This is a telescoping sum! This gives us

$$U(f,P) - L(f,P) = \frac{1}{n}(f(1) - f(0)) < \varepsilon.$$

Therefore f is integrable.

Extra Space