MA 575 Midterm Exam.

October 6 2017

Name: ____

Problem 1. *Suppose* $X \subset \mathcal{C}$ *has a supremum. Show that* $\sup(X) = \sup(\overline{X})$*.*

Proof: Let $u = \sup X$. Suppose $y > u$. Then take $v > y$ and consider the open interval (u, v). This contains y but since u is an upper bound for X, it does not contain any $x \in X$. Therefore y is not a limit point of X.

Therefore if y is a limit point of X then $y < u$, and moreover since u is an upper bound for X, if $x \in X$ then $x < u$. Therefore $x < u$ for all $x \in \overline{X}$. Thus u is an upper bound of $\overline{X}$.

Now if $z < u$ then it follows from the definition of supremum that z is not an upper bound of X, so z is not an upper bound of $\overline{X}$. Therefore any upper bound of $\overline{X}$ is greater than or equal to u, and hence u is the supremum of $\overline{X}$.

Problem 2. *Suppose $f : \mathcal{C} \to \mathcal{C}$ such that $f(X)$ is connected for each $X \subset \mathcal{C}$. Does it follow that f is continuous? Justify your answer.*

Solution: Yes, f must be continuous – in fact it must be constant. To prove this, consider any $a, b \in \mathcal{C}$ and set $X = \{a, b\}$.

Then $f(X) = \{f(a), f(b)\}$ is connected. But if $f(a) \neq f(b)$ then take z between $f(a)$ and $f(b)$. The sets $(-\infty, z)$ and (z, ∞) are disjoint, open, and contain $f(X)$ in their union. Moreover one of these sets contains $f(a)$ and the other contains $f(b)$, so they disconnect $f(X)$. This contradicts the hypothesis, so we must have $f(a) = f(b)$.

But a and b were arbitrary, so $f(a) = f(b)$ for all $a, b \in \mathcal{C}$. Thus f is constant and hence continuous.

Problem 3. *Suppose X is compact and $f : \mathcal{C} \to \mathcal{C}$ is continuous. Show that $f(X)$ is compact.*

Proof: Suppose $\mathcal{U} = \{U_\lambda\}$ is an open cover of $f(X)$. Define $\mathcal{V} = \{f^{-1}(U_\lambda)\}$. Since f is continuous and eachU_λ is open, every $f^{-1}(U_\lambda)$ is open. Moreover since $f(X) \subseteq \cup_\lambda U_\lambda$ then it follows from set relations that

$$X \subseteq f^{-1}(f(X)) \subseteq f^{-1}(\cup_\lambda U_\lambda) = \cup_\lambda f^{-1}(U_\lambda).$$

Therefore $\mathcal{V}$ is an open cover of X. Since X is compact, there is a finite subcover $\{f^{-1}(U_{\lambda_1}), \ldots, f^{-1}(U_{\lambda_n})\}$ which also covers X. Then $X \subseteq \cup_{j=1}^n f^{-1}(U_{\lambda_j})$, so it follows from set relations that

$$f(X) \subseteq f(\cup_{j=1}^n f^{-1}(U_{\lambda_j})) = \cup_{j=1}^n f(f^{-1}(U_{\lambda_j})) \subseteq \cup_{j=1}^n U_{\lambda_j}.$$

Therefore $\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$ is a finite subset of $\mathcal{U}$ that covers $f(X)$. Since $\mathcal{U}$ was arbitrary, it follows that every open cover of $f(X)$ has a finite subcover, so $f(X)$ is compact.

Problem 4. *We define X to be dense in $\mathcal{C}$ if every nonempty open set in $\mathcal{C}$ contains some $x \in X$.*

Suppose X is dense in $\mathcal{C}$ and $f : \mathcal{C} \to \mathcal{C}$ is continuous, with $f(x) = c$ for all $x \in X$. Show that $f(x) = c$ for all $x \in \mathcal{C}$.

Proof: Let $U = \mathcal{C} \setminus \{c\}$. Since U is open, and f is continuous, $f^{-1}(U)$ is open.

But if $x \in X$, then $f(x) = c \notin U$ so $x \notin f^{-1}(U)$. Therefore $f^{-1}(U)$ is an open set which does not contain any point in X, and hence $f^{-1}(U)$ must be empty. But then no point in $\mathcal{C}$ can map to any element in U so $f(x) = c$ for all $x \in \mathcal{C}$.

Problem 5. *Suppose* $\{[a_j, b_j]\}_{j\in\mathbb{N}}$ *is a collection of non-empty closed intervals such that* $[a_{j+1}, b_{j+1}] \subseteq [a_j, b_j]$. *Show that there exists* x *such that* $x \in \cap_{j\in\mathbb{N}}[a_j, b_j]$.

Proof: We begin with a claim: for any $i, j \in \mathbb{N}$, we have $a_i \leq b_j$. To prove the claim, consider two cases. First, if $i \leq j$, then $a_i \leq a_j \leq b_j$. Second, if $i > j$ then $a_i \leq b_i \leq b_j$.

Now let $A = \{a_i\}_{i\in\mathbb{N}}$. By the claim, any b_j is an upper bound for A. Therefore A is nonempty and bounded above, and hence it has a supremum which we will denote a.

Since a is an upper bound for A, $a \geq a_j$ for all $j \in \mathbb{N}$. Moreover, by the claim, each b_j is an upper bound for A, so $a = \sup A \leq b_j$ for all $j \in \mathbb{N}$. Therefore $a \in [a_j, b_j]$ for all $j \in \mathbb{N}$, and hence $a \in \cap_{j\in\mathbb{N}}[a_j, b_j]$.

Extra Space