1. The Natural Numbers

Definition 1 (Definition of $\mathbb{N}$)**.** *The natural numbers are a set $\mathbb{N}$, together with a successor function $S : \mathbb{N} \to \mathbb{N}$, which satisfies the following axioms:*

Axiom 1: *: $\mathbb{N}$ is nonempty, and contains an object called 0.*
Axiom 2: *: If $n, m \in \mathbb{N}$, and $n \neq m$, then $S(n) \neq S(m)$.*
Axiom 3: *: There is no $n \in \mathbb{N}$ such that $0 = S(n)$.*
Axiom 4: *: If $A \subset \mathbb{N}$, and $0 \in A$, and for every $a \in A$, $S(a) \in A$, then $A = \mathbb{N}$.*

These puppies should have their own names.

Definition 2.

$$\begin{aligned} 1 &= S(0) \\ 2 &= S(1) \\ 3 &= S(2) \\ 4 &= S(3) \text{ etc.} \end{aligned}$$

Some silly but important propositions follow.

Proposition 1.1. $4 \neq 0$.

Proposition 1.2. $1 \neq 2$.

The following requires Axiom 4.

Proposition 1.3. *Suppose $n \in \mathbb{N}$. Then $n \neq S(n)$.*

2. Arithmetic

We start with a recursion lemma: it tells us we can define functions recursively.

Lemma 2.1. *Suppose $f : \mathbb{N} \to \mathbb{N}$, and $c \in \mathbb{N}$. There exists a unique function $g : \mathbb{N} \to \mathbb{N}$ such that*

$$g(0) = c \text{ and } g(S(n)) = f(g(n)).$$

This is of enormous help in defining $n + m$ for all natural numbers n, m simultaneously.

Definition 3. *Suppose $m \in \mathbb{N}$. We define $0 + m = m$. Moreover if we have defined $n + m$, then we define $S(n) + m = S(n + m)$.*

Another silly proposition:

Proposition 2.2. $2 + 2 = 4$.

We want to show that our definition of addition behaves the way we secretly want it to behave. To start, we show that addition is commutative.

Lemma 2.3. *Let $n \in \mathbb{N}$. Then $n + 0 = n$.*

Lemma 2.4. *Let $n, m \in \mathbb{N}$. Then $n + S(m) = S(n + m)$.*

Proposition 2.5. *Suppose* $n, m \in \mathbb{N}$. *Then* $n + m = m + n$.

Similarly we can prove associativity:

Proposition 2.6. *Let* $n, m, p \in \mathbb{N}$. *Then* $n + (m + p) = (n + m) + p$.

Also important is the cancellation law:

Proposition 2.7. *Suppose* $n, m, p \in \mathbb{N}$, *and* $n + m = n + p$. *Then* $m = p$.

While we're here we might as well spill the beans on what S is:

Proposition 2.8. *Let* $n \in \mathbb{N}$. *Then* $S(n) = n + 1$.

We can go on in this vein to define order ($a < b$), multiplication, subtraction (though this makes more sense in $\mathbb{Z}$ – in fact you can use this to define $\mathbb{Z}$), etc. Some of this will be on your problem set but for the most part we're going to skip ahead past $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ and move on to $\mathbb{R}$...