1. Problem Set 10

Problem 1. *Suppose* $\{a_n\}$ *converges to* L *and* $f : \mathbb{R} \to \mathbb{R}$ *is continuous. Show that* $\{f(a_n)\}$ *converges to* $f(L)$.

Problem 2. *Define* $f : \mathbb{R} \to \mathbb{R}$ *to be a contraction map if there exists* $c \in (0, 1)$ *such that*

$$|f(x) - f(y)| < c|x - y|$$

for all $x, y \in \mathbb{R}$. *Show that if* f *is a contraction map then for any* $x \in \mathbb{R}$, *the sequence*

$$x, f(x), f(f(x)), f(f(f(x))), \ldots$$

converges.

Problem 3. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and a contraction map. Show that there exists* $x \in \mathbb{R}$ *such that* $f(x) = x$.

Problem 4. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is positive, continuous, and nonincreasing, and* $\{a_n\}$ *is the sequence given by* $a_n = f(n)$ *for all* $n \in \mathbb{N}$. *Show that*

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(t)dt < \infty.$$

Problem 5. *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are positive sequences, and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0.$$

Show that $\{a_n\}$ *is summable if and only if* $\{b_n\}$ *is summable.*

Problem 6. *Suppose* $\{a_n\}$ *is a positive nonincreasing sequence which converges to zero. Show that*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges. Hint: consider the subsequence of partial sums $s_2, s_4, s_6, \ldots$ *and the subsequence of partial sums* $s_1, s_3, s_5, \ldots$ *separately – show that one of these subsequences is increasing, another is decreasing, and obtain a relationship between the subsequences.*

Problem 7. *A sequence* $\{a_n\}$ *is defined to be Cesaro summable with Cesaro sum* L *if the partial sums* $\{s_n\}$ *have the property that*

$$\lim_{n\to\infty} \frac{s_1 + \ldots + s_n}{n} = L.$$

Show that if $\{a_n\}$ *is summable with sum* L, *then* $\{a_n\}$ *is Cesaro summable with Cesaro sum* L. *Show that the converse does not hold: there exists a sequence which is Cesaro summable but not summable.*

Problem 8. *Suppose* $\{a_n\}$ *is summable but not absolutely summable. Define the sequences* $\{a_n^+\}$ *and* $\{a_n^-\}$ *by*

$$a_n^{\pm} = \begin{cases} a_n & \text{if } \pm a_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that neither $\{a_n^+\}$ *nor* $\{a_n^-\}$ *are summable.*

Problem 9 (Optional)**.** *Suppose* $\{a_n\}$ *is summable but not absolutely summable. Show that for any* $L \in \mathbb{R}$*, there is a rearrangement of* $\{a_n\}$ *whose sum converges to* L*. Hint: if* $L > 0$ *then try to add terms of* $\{a_n^+\}$ *until you are greater than* L *and then add terms of* $\{a_n^-\}$ *until you are less than* L*. Try to convince yourself that this process makes sense and gives a sum that converges to* L*!*