1. Problem Set 6

Problem 1. *Suppose*

$$\lim_{x\to p} f(x) = L \qquad \text{and} \qquad \lim_{x\to p} g(x) = M.$$

Show that

$$\lim_{x\to p} f(x)g(x) \quad = LM,$$

$$\text{and} \quad \lim_{x\to p} \frac{1}{g(x)} \qquad = \tfrac{1}{M} \quad \text{if } M \neq 0.$$

Problem 2. *Suppose*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \neq \mathbb{Q}. \end{cases}$$

Show that $\lim_{x\to a} f(x)$ *does not exist, for any* $a \in \mathbb{R}$.

Problem 3. *Suppose* $f(x) \leq g(x) \leq h(x)$ *for all* $x \in \mathbb{R}$ *and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Show that $\lim_{x\to a} g(x) = L$.

Problem 4. *In the spirit of the* $\varepsilon - \delta$ *definitions of* $\lim_{x\to a} f(x) = L$, *give definitions for*

$$\lim_{x\to\infty} f(x) = L, \qquad \lim_{x\to a} f(x) = \infty, \qquad \text{and } \lim_{x\to a^+} f(x) = L.$$

Problem 5. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is continuous, and* $f(p) > 0$. *Use the epsilon-delta definition of continuity to show that there exists an open interval* (a, b) *containing* p *such that* $f(x) > 0$ *for all* $x \in (a, b)$. *(Optional: do the same thing but using the open sets definition of continuity. Which is easier?)*

Problem 6. *Show that if* $y \geq 0$ *then there exists a unique* $x \geq 0$ *such that* $x^2 = y$. *Use this to define the square root function, and show that the square root function is continuous at* x *for every* $x > 0$.

Problem 7. *A function* $f : A \to \mathbb{R}$ *is Lipschitz if there exists* $C > 0$ *such that for all* $x, y \in A$

$$|f(x) - f(y)| \leq C|x - y|.$$

Let A *be open and suppose* $f : A \to \mathbb{R}$ *is Lipschitz. Show that* f *is continuous at all* $x \in A$.