1. Problem Set 7

Problem 1. *Suppose f is differentiable on (a,b). Show that f is nondecreasing on (a,b) if and only if $f'(x) \geq 0$ on (a,b).*

Problem 2. *Suppose $S : \mathbb{R} \to [-1,1]$ is a differentiable function such that $S'(n\pi) = 1$ whenever n is an even integer and $S'(n\pi) = -1$ whenever n is an odd integer. (You should think of $S(x) = \sin(x)$, but strictly speaking we haven't defined $\sin \ldots$ or π.) Show that the function*

$$f(x) = \begin{cases} x^2 S(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable everywhere, but its derivative is not continuous at zero.

Problem 3. *Suppose that f is continuous at a and $f'(x)$ exists for all $x \neq a$. Suppose also that $\lim_{x \to a} f'(x)$ exists. Show that $f'(a)$ exists and*

$$f'(a) = \lim_{x \to a} f'(x).$$

Convince yourself that this means that the only way the derivative can fail to be continuous at a is if $\lim_{x \to a} f'(x)$ does not exist.

Problem 4. *Suppose $f'(x) = 0$. Show that f has a local maximum at x if $f''(x) > 0$, and f has a local minimum at x if $f''(x) < 0$.*

Problem 5. *Suppose $f'(x) \geq 4$ for all $x \in [0,1]$. Show that there is an interval of length $1/4$ on which $|f(x)| > 1$.*

Problem 6. *Suppose $f : (a,b) \to \mathbb{R}$ is one-to-one and continuous. Show that f is either increasing or decreasing on (a,b).*

Problem 7. *Suppose $f : (a,b) \to \mathbb{R}$ is one-to-one and continuous. Show that f^{-1} is continuous.*

Problem 8. *Suppose f is continuous, differentiable, and one-to-one on (a,b). Show that f^{-1} is differentiable at all $x \in (a,b)$ such that $f'(f^{-1}(x)) \neq 0$, and*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Hint: You must do more than naively use the chain rule: the chain rule only applies if you already know that f^{-1} is differentiable.