1. Problem Set 9

Problem 1. *Suppose f is integrable on $[a,b]$. Show that $|f|$ is integrable and*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$$

Problem 2. *Suppose f, g are continuous on $[a,b]$. Show that*

$$\left(\int_a^b f(t)g(t)dt\right)^2 \leq \left(\int_a^b f^2(t)dt\right)\left(\int_a^b g^2(t)dt\right).$$

(Hint: if the left side is not zero, set $F = f/\sqrt{\int_a^b f^2(t)dt}$, define G similarly, and consider the quantity $(F-G)^2$).

Problem 3. *Suppose f is integrable on $[a,b]$ and*

$$F(x) = \int_a^x f(t)dt.$$

Show that F is continuous at all $x \in (a,b)$.

Problem 4. *Suppose f is continuous on $[0,1]$. Show that*

$$\lim_{n\to\infty} \sum_{k=1}^n f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx.$$

Problem 5. *Suppose $\{a_n\}$ is a nondecreasing sequence which is bounded above. Show that $\{a_n\}$ converges.*

Definition 1. *Suppose $\{a_n\}$ is a bounded sequence. Define new sequences*

$$u_n = \sup\{a_m | m > n\} \quad \text{and} \quad l_n = \inf\{a_m | m > n\}$$

We say that

$$\limsup a_n = p \text{ if } \lim_{n\to\infty} u_n = p$$

and

$$\liminf a_n = p \text{ if } \lim_{n\to\infty} l_n = p.$$

Problem 6. *Suppose $\{a_n\}$ is a bounded sequence. Show that $\limsup a_n$ and $\liminf a_n$ always exist.*

Problem 7. *Suppose $\{a_n\}$ is a bounded sequence, and $\limsup a_n = \liminf a_n = p$. Show that $\{a_n\}$ converges to p.*

Problem 8. *Suppose $A \subset \mathbb{R}$. Show that $p \in \overline{A}$ if and only if there exists a sequence $\{a_n\}$ such that each $a_n \in A$, and $\{a_n\}$ converges to p.*