1. Sequences and Subsequences

Definition 1.1. *A* sequence *is a function $a : N \to \mathbb{R}$ from the natural numbers to the real numbers.*

By setting $a_n = a(n)$, we think of a sequence a as a list $a_1, a_2, a_3 \dots$ of real numbers. We use the notation $\{a_n\}_{n=1}^{\infty}$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply $\{a_n\}$. More generally, we also use the term sequence to refer to a function defined on $\{n \in N | n \geq n_0\}$ for any fixed $n_0 \in \mathbb{N}$. We denote this by writing $\{a_n\}_{n=n_0}^{\infty}$ for such a sequence.

Definition 1.2. *We say that a sequence $\{a_n\}$* converges *to a point $p \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $|a_n - p| < \varepsilon$.*

If $\{a_n\}$ converges to p, we write this as:

$$\lim_{n\to\infty} a_n = p,$$

and call p the limit *of $\{a_n\}$. If $\{a_n\}$ does not converge to any point p, we call it* divergent*.*

Theorem 1.1. *Suppose that*

$$\lim_{n\to\infty} a_n = p \quad \text{and} \quad \lim_{n\to\infty} a_n = p'.$$

Then $p = p'$. In other words, limits of sequences are unique.

Definition 1.3. *Let (a_n) be a sequence. A* subsequence *of $\{a_n\}$ is a sequence b defined by the composition $b = a \circ \eta \colon \mathbb{N} \to \mathbb{R}$, where $\eta \colon \mathbb{N} \to \mathbb{N}$ is an increasing function.*

Remark: By increasing, we mean that η has the property that if $n < m$, then $\eta(n) < \eta(m)$. An increasing function on the natural numbers has the property that $\eta(k) \geq k$.

Note that η itself defines a sequence $n_k = \eta(k)$, so we usually write $b_k = a_{n_k}$.

Exercise 1.2. *Construct a divergent sequence with a subsequence which converges.*

Theorem 1.3. *If $\{a_n\}$ converges to p, then so do its subsequences.*

Sometimes divergent sequences have points that behave like limits, but are not necessarily unique:

Definition 1.4. *A point $p \in \mathbb{R}$ is an* accumulation point *of $\{a_n\}$ if for all $\varepsilon > 0$ and $M \in \mathbb{N}$, there exists $n > M$ such that $|a_n - p| < \varepsilon$.*

Remark: Equivalently, we could say that $p \in \mathbb{R}$ is an accumulation point of $\{a_n\}$ if for all $\varepsilon > 0$, there exist infinitely many n such that $|a_n - p| < \varepsilon$.

Exercise 1.4. *Construct a sequence with two distinct accumulation points. Construct a sequence with infinitely many accumulation points. Construct a sequence with no accumulation points.*

Proposition 1.5. *Let $\{a_n\}$ be a sequence and suppose that there exists a subsequence $(b_k = a_{n_k})$ that converges to p. Then p is an accumulation point of (a_n).*

Theorem 1.6. *A point p is an accumulation point of $\{a_n\}$ if and only if there exists a subsequence b_k converging to p.*

Corollary 1.7. *Suppose that $\lim_{n\to\infty} a_n = p$. Then p is the only accumulation point of the sequence $\{a_n\}$.*

Definition 1.5. *A sequence $\{a_n\}$ is bounded if the set of all a_n is bounded. Similar definitions apply for bounded above and bounded below.*

Theorem 1.8. *Suppose $\{a_n\}$ converges. Then $\{a_n\}$ is bounded.*

Proposition 1.9 (Monotone Convergence Theorem*)**.** *Suppose $\{a_n\}$ is nondecreasing (meaning that $a_n \leq a_{n+1}$ for each n) and bounded above. Then $\{a_n\}$ converges.*

Theorem 1.10 (Bolzano-Weierstrass*)**.** *Every bounded sequence has a convergent subsequence.*

Theorem 1.11. *Suppose $\{a_n\}$ converges to L and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then $\{f(a_n)\}$ converges to $f(L)$.*

2. Cauchy Sequences

Definition 2.1. *A sequence $\{a_n\}$ is Cauchy if for all $\varepsilon > 0$ there exists N such that for all $n, m > N$,*

$$|a_n - a_m| < \varepsilon.$$

Proposition 2.1. *If $\{a_n\}$ converges then it is Cauchy.*

Lemma 2.2. *Suppose $\{a_n\}$ is Cauchy and a subsequence of $\{a_n\}$ converges to p. Then $\{a_n\}$ converges to p.*

Lemma 2.3. *If $\{a_n\}$ is Cauchy then $\{a_n\}$ is bounded.*

Theorem 2.4 (*)**.** *A sequence $\{a_n\}$ is Cauchy if and only if it converges.*

3. Series

Definition 3.1. *Consider a sequence $\{a_n\}$. We define the n^{th} partial sum of $\{a_n\}$ by*

$$s_n = a_1 + \ldots + a_n.$$

We say that $\{a_n\}$ is summable (or $\displaystyle\sum_{n=1}^{\infty} a_n$ converges) if $\{s_n\}$ converges, and then we define

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Exercise 3.1. *Prove that if $\{a_n\}$ and $\{b_n\}$ are summable then so is $\{a_n + b_n\}$, and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Theorem 3.2 (Vanishing Criterion)**.** *If $\{a_n\}$ is summable then $\lim_{n\to\infty} a_n = 0$. Note that the converse is false!*

Proposition 3.3 (Boundedness Criterion)**.** *Suppose $\{a_n\}$ is nonnegative and the sequence of its partial sums $\{s_n\}$ is bounded. Then $\{a_n\}$ is summable.*

Theorem 3.4 (Comparison Test)**.** *Suppose $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$, and $\{b_n\}$ is summable. Then $\{a_n\}$ is summable.*

Lemma 3.5. *The geometric series $\sum_{n=1}^{\infty} r^n$ converges if $0 \le r < 1$ and diverges if $r \ge 1$.*

Theorem 3.6 (Ratio Test*)**.** *Suppose $0 \le a_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$. Then $\{a_n\}$ is summable if $r < 1$ and not summable if $r > 1$. If $r = 1$ then $\{a_n\}$ may or may not be summable.*

Theorem 3.7 (Integral Test)**.** *Suppose f is positive, continuous, and nonincreasing, and $a_n = f(n)$. Then $\{a_n\}$ is summable if and only if*

$$\lim_{x\to\infty} \int_1^x f(t)dt$$

exists.

Definition 3.2. *We say that $\{a_n\}$ is absolutely summable, or $\sum_{n=1}^{\infty} a_n$ converges absolutely, if $\{|a_n|\}$ is summable.*

Theorem 3.8. *Suppose $\sum_{n=1}^{\infty} a_n$ converges absolutely. Then $\sum_{n=1}^{\infty} a_n$ converges.*

Remark: The alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \ldots$ is converges but not absolutely. However, it is possible to show that such a series can be rearranged to converge to any number you want – which is terribly depraved behavior. Absolutely convergent series are too good for these problems.

Theorem 3.9. *Suppose $\{a_n\}$ is absolutely summable, and $\{b_n\}$ is a rearrangement of $\{a_n\}$ (i.e. $b_n = a_{f(n)}$ for some bijective function $f : \mathbb{N} \to \mathbb{N}$.) Then $\{b_n\}$ converges absolutely, and*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n.$$

4. Taylor Series

Theorem 4.1. *Suppose f is differentiable $n+1$ times, and $f^{(n+1)}$ is continuous. Then for any $a \in \mathbb{R}$,*

$$f(x) = \sum_{k=1}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + R_{n,a}$$

where

$$R_{n,a} = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Theorem 4.2. *The expression*

$$R_{n,a}(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$$

has the properties that

- $R_{n,a}(x) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n(x-a)$ *for some* $t \in (a,x)$
- $R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$ *for some* $t \in (a,x)$.

Remark: For many well behaved functions like e^x, $\sin(x)$, etc. it is easy to show that $\lim_{n\to\infty} R_{n,a}(x) = 0$ for any x. This gives us the classic Taylor *series*

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

For other functions it is possible to obtain series expressions for limited values of x. For example, for $|x| < 1$, the following series expressions hold:

$$\begin{aligned} \text{arctan}x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \dots \end{aligned}$$

Note that the last one we already knew: it's the sum of a geometric series!

5. Sequences of Functions

Definition 5.1. *Suppose* $f_n, f : A \to \mathbb{R}$. *We say that* f_n *converge to* f *pointwise on* A *if*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in A$.

Definition 5.2. *Suppose $f_n, f : A \to \mathbb{R}$. We say that f_n converge to f uniformly on A if for every $\varepsilon > 0$ there exists N such that for all $n > N$ and $x \in A$,*

$$|f_n(x) - f(x)| < \varepsilon.$$

Exercise 5.1. *Give an example of a sequence of functions that converges pointwise on $[a, b]$ but not uniformly.*

Theorem 5.2 (*)**.** *Suppose $f_n : A \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly on A. Then f is continuous on A.*

Theorem 5.3. *Suppose $f_n, f : [a, b] \to \mathbb{R}$ are integrable and $f_n \to f$ uniformly on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Theorem 5.4. *Suppose $f_n, f : (a, b) \to \mathbb{R}$, $f_n \to f$ pointwise, and each f_n is differentiable, and $f_n' \to f$ uniformly on (a, b). Then f is differentiable on $[a, b]$, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

for all $x \in (a, b)$.