1. The Mean Value Theorem

Exercise 1.1. *Prove* Rolle's Theorem*: If f is continuous at all $x \in [a, b]$, and f is differentiable at all $x \in (a, b)$, and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Exercise 1.2. *Prove the* Mean Value Theorem*: If f is continuous at all $x \in [a, b]$ and differentiable at all $x \in (a, b)$, then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Exercise 1.3. *Prove that $f'(x) = 0$ for all $x \in (a, b)$ if and only if f is constant on (a, b).*

Exercise 1.4. *Prove the* Cauchy Mean Value Theorem*: if f and g are continuous at all $x \in [a, b]$, and differentiable at all $x \in (a, b)$, then there exists $c \in (a, b)$ such that*

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

Exercise 1.5. *Prove (the following version of) L'Hôpital's Rule: if f and g are differentiable at a, $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, and $\lim_{x \to a} f'(x)/g'(x)$ exists, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$