1. Worksheet Oct 2

Recall that there is an injective map $I : \mathbb{Q} \to \mathbb{R}$ which respects the ordered field structures:

$$\begin{aligned} I(a) + I(b) &= I(a+b) \\ I(a) \cdot I(b) &= I(ab) \\ a < b &\Leftrightarrow I(a) < I(b) \end{aligned}$$

We identify $\mathbb{Q}$ and its subsets with their image under this map. Last class we proved the proposition

Proposition 1.1. *$\mathbb{Z} \subset \mathbb{R}$ is not bounded.*

Exercise 1.2. *Prove that $\mathbb{Z} \subset \mathbb{R}$ is closed.*

Exercise 1.3. *Suppose that $x, y \in \mathbb{R}$ and $y - x > 1$. Show that there exists $n \in \mathbb{Z}$ such that $x < n < y$.*

Definition 1.1. *We say that $X \subset \mathbb{R}$ is dense in $\mathbb{R}$ if every open interval $(a, b) \subset \mathbb{R}$ contains an element of X.*

Exercise 1.4. *Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.*

Exercise 1.5. *Show that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. (You can use the fact that $\sqrt{2}$ is irrational, without proof.)*