1. Worksheet Oct 30

Definition 1. *A partition P of $[a,b]$ is a finite collection of points $t_0, \ldots, t_n$ such that*

$$a = t_0 < \ldots < t_n = b.$$

Definition 2. *Suppose f is bounded on $[a,b]$, and P is a partition of $[a,b]$. The upper and lower sums of f on $[a,b]$ with respect to P are the quantities*

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{and} \quad L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

where M_i and m_i are the supremum and infimum, respectively, of the sets

$$\{f(x) | x \in [t_{i-1}, t_i]\}.$$

Definition 3. *Suppose f is bounded on $[a,b]$. We say f is (Riemann) integrable on $[a,b]$ if*

$$\inf_P U(f,P) = \sup_P L(f,P),$$

and define this quantity to be the integral

$$\int_a^b f(x)dx.$$

Last time we showed the theorem

Theorem 1.1. *Let P_1, P_2 be partitions of $[a,b]$. Then*

$$L(f,P_1) \leq U(f,P_2).$$

The theorem implies that $\inf_P U(f,P) \geq \sup_P L(f,P)$ for any bounded function f.

Exercise 1.2. *Show that $f(x) = x^2$ is integrable on $[0,1]$.*

Exercise 1.3. *Suppose f is bounded on $[a,b]$. Show that f is Riemann integrable on $[a,b]$ if and only if for all $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon.$$

Exercise 1.4. *Show that if f is continuous on $[a,b]$ then f is integrable on $[a,b]$.*

Exercise 1.5. *Suppose f is integrable on $[a,b]$ and on $[b,c]$. Show that f is integrable on $[a,c]$, and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$