1. Worksheet Sep 8

Definition 1.1. *Suppose $X \subseteq \mathcal{C}$. We say X is disconnected if there exist open sets $A, B \subset \mathcal{C}$ such that*

$$\begin{aligned} X &\subseteq A \cup B \\ A \cap B &= \varnothing \\ A \cap X, B \cap X &\neq \varnothing. \end{aligned}$$

We say X is connected if it is not disconnected.

Exercise 1.1. *Suppose $x, y \in \mathcal{C}$, and $x < y$. Show that there exists a $z \in \mathcal{C}$ such that $x < z < y$.*

Exercise 1.2. *Let (a, b) be a nonempty open interval. Convince yourself that (a, b) is infinite.*

Exercise 1.3. *Suppose $x \in \mathcal{C}$. Show that x is a limit point of $\mathcal{C}$.*

Exercise 1.4. *Let $[a, b]$ be a nonempty closed interval. Show that $[a, b]$ is connected.*

Proof. We begin by showing that $(-\infty, b]$ is connected. Suppose $(-\infty, b]$ is disconnected. Then there exist open sets A and B disconnect $(-\infty, b]$. Without loss of generality $b \in B$. Define $A' = A \cap (-\infty, b)$ and $B' = B \cup (b, \infty)$. By theorems 3.7 and 3.10, A' and B' are open. One can check that

$$A' \cap B' = A \cap B = \varnothing$$

and $A, B \neq \varnothing$. Moreover

$$A \cup B = \mathcal{C}$$

so these sets disconnect the continuum, which contradicts the connectedness axiom. Therefore $(-\infty, b]$ is connected.

Now suppose $[a, b]$ is disconnected. Then there exist open sets U and V which disconnect $[a, b]$. Without loss of generality, $a \in U$. Then let $U' = U \cup (-\infty, a)$ and $V' = V \cap (a, \infty)$. As above, one can check that U', V' are open, $U' \cap V' = \varnothing$, and $U' \cap (-\infty, b], V' \cap (-\infty b] \neq \varnothing$. Moreover $(-\infty, b] \subseteq U' \cup V'$, so U' and V' disconnect $(-\infty, b]$. But this contradicts the first part of the proof, and hence $[a, b]$ is connected. □

Exercise 1.5. *Let (a, b) be a nonempty open interval. Show that (a, b) is connected.*