Math 633 Midterm

March 3 2016

Do **three** of the following five questions. If you complete more than three questions, indicate clearly which three you would like to be graded.

The first two questions cannot be done outside of class.

(1) Show that there exists $C > 0$ such that

$$|u(0)| \leq C\|u\|_{W^{1,1}(\mathbb{R})}$$

for any $u \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$. Explain how this inequality can be used to extend the operator $T : u \mapsto u(0)$ to a bounded linear operator $T : W^{1,1}(\mathbb{R}) \to \mathbb{R}$ with the bound

$$|T(u)| \leq C\|u\|_{W^{1,1}(\mathbb{R})}.$$

(2) Suppose $u \in W^{k,p}(\mathbb{R}^n)$ and $\eta \in C_c^\infty(\mathbb{R}^n)$. Let $|\alpha| \leq k$. Show that

$$D^\alpha(\eta * u) = \eta * D^\alpha u,$$

where D^α here indicates the weak derivative.

(3) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Consider the boundary value problem

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

(a) State what it means for $u \in H_0^1(\Omega)$ to be a weak solution of the boundary value problem.

(b) Suppose $f \in L^{\frac{2n}{n+2}}(\Omega)$, then show there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem above.

(4) Let Ω be a smooth bounded domain and $b \in \mathbb{R}^n$ be a fixed vector. Show that there exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C\|b \cdot \nabla u\|_{L^2(\Omega)}$$

for all $u \in H_0^1(\Omega)$. Explain why this doesn't hold for all $u \in H^1(\Omega)$.

(5) Let L be the differential operator defined by $Lu = b \cdot \nabla u + cu$, where b and c are constant. Suppose that for any domain Ω of the form

$$\Omega = \{x \in \mathbb{R}^n | x_n > f(x_1, \ldots, x_{n-1})\},$$

where f is smooth, there exists a constant $C_\Omega > 0$ independent of b such that the inequality

$$\|u\|_{L^2(\Omega)} \leq C_\Omega(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)})$$

holds for all $u \in C^1(\Omega) \cap H^1(\Omega)$. Show then that for any smooth bounded domain Ω, a similar inequality holds provided that b is sufficiently small.