MATH 633 MIDTERM

Answers

(1) Show that there exists $C > 0$ such that

$$|u(0)| \leq C\|u\|_{W^{1,1}(\mathbb{R})}$$

for any $u \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$. Explain how this inequality can be used to extend the operator $T : u \mapsto u(0)$ to a bounded linear operator $T : W^{1,1}(\mathbb{R}) \to \mathbb{R}$ with the bound

$$|T(u)| \leq C\|u\|_{W^{1,1}(\mathbb{R})}.$$

Proof: Let $u \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$. Then the fundamental theorem of calculus implies that

$$u(0) = \int_{-\infty}^{0} u'(x)\,dx,$$

since $u \to 0$ as $x \to \infty$. Taking absolute values and applying basic inequalities, we get that

(1) $$|u(0)| \leq \int_{-\infty}^{\infty} |u'(x)|\,dx \leq \|u\|_{W^{1,1}(\mathbb{R})}.$$

Now if $v \in W^{1,1}(\mathbb{R})$, then there exists a sequence $\{v_k\} \subset C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ such that $\|v_k - v\|_{W^{1,1}(\mathbb{R})} \to 0$. Then the sequence $\{v_k\}$ is Cauchy in $W^{1,1}(\mathbb{R})$, and so the inequality (1) implies that the sequence $v_k(0)$ is Cauchy in $\mathbb{R}$. We can define

$$Tv = \lim_{k\to\infty} v_k(0).$$

This is well defined, since if w_k is another sequence in $C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ which converges to v, then (1) implies that

$$|v_k(0) - w_k(0)| \leq \|v_k - w_k\|_{W^{1,1}(\mathbb{R})}$$

and so as k goes to infinity, $|v_k(0) - w_k(0)| \to 0$.

Moreover for any $\varepsilon > 0$, there exists k such that

$$|Tv| \leq |v_k(0)| + \varepsilon \leq \|v_k\|_{W^{1,1}(\mathbb{R})} + \varepsilon \leq \|v\|_{W^{1,1}(\mathbb{R})} + 2\varepsilon,$$

and so

$$|Tv| \leq \|v\|_{W^{1,1}(\mathbb{R})}$$

as required.

(2) Suppose $u \in W^{k,p}(\mathbb{R}^n)$ and $\eta \in C_c^\infty(\mathbb{R}^n)$. Let $|\alpha| \leq k$. Show that

$$D^\alpha(\eta * u) = \eta * D^\alpha u,$$

where D^α here indicates the weak derivative.

Proof: Let u, η, and α be as specified. Since $\eta \in C_c^\infty(\mathbb{R}^n)$, we know that $\eta * u$ is smooth, and thus $D^\alpha(\eta * u)$ is a classical derivative. Now we have

$$\begin{aligned} D^\alpha(\eta * u)(x) &= D^\alpha \int_{\mathbb{R}^n} \eta(x-y)u(y)dy \\ &= \int_{\mathbb{R}^n} D_x^\alpha \eta(x-y)u(y)dy \end{aligned}$$

with the switching of limits justified since $\eta \in C_c^\infty(\mathbb{R}^n)$. Then

$$D^\alpha(\eta * u)(x) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_y^\alpha \eta(x-y)u(y)dy$$

and now it follows from the definition of the weak derivative that

$$\begin{aligned} D^\alpha(\eta * u)(x) &= \int_{\mathbb{R}^n} \eta(x-y)D_y^\alpha u(y)dy \\ &= \eta * D^\alpha u(x) \end{aligned}$$

(3) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Consider the boundary value problem

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

(a) State what it means for $u \in H_0^1(\Omega)$ to be a weak solution of the boundary value problem.

(b) Suppose $f \in L^{\frac{2n}{n+2}}(\Omega)$, then show there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem above.

Answer: $u \in H_0^1(\Omega)$ is a weak solution to the boundary value problem if

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

for every $v \in H_0^1(\Omega)$ (or every $v \in C_c^\infty(\Omega)$: there are other equivalent statements). As noted during the exam, the rest of this question requires $n \geq 3$.

Suppose $f \in L^{\frac{2n}{n+2}}(\Omega)$. Then for $v \in H_0^1(\Omega)$, Hölder's inequality gives us

$$\begin{aligned} \left|\int_\Omega fv\,dx\right| &\leq \int_\Omega |fv|\,dx \\ &\leq \|f\|_{L^{\frac{2n}{n+2}}(\Omega)} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \end{aligned}$$

Then Sobolev embedding (Gagliardo-Nirenberg-Sobolev) tells us that

$$\|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C\|v\|_{H^1(\Omega)}$$

so

$$\left|\int_\Omega fv\,dx\right| \leq C\|f\|_{L^{\frac{2n}{n+2}}(\Omega)} \|v\|_{H^1(\Omega)}.$$

Therefore the linear functional $v \mapsto \int_\Omega fv\,dx$ is bounded on $H_0^1(\Omega)$. Now as we noted in class, Poincaré's inequality implies that

$$B[u,v] = \int_\Omega \nabla u \cdot \nabla v\,dx$$

is an inner product on H_0^1, so the Riesz representation theorem says that there exists $u \in H_0^1$ such that

$$v \mapsto \int_\Omega fv\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx$$

for all $v \in H_0^1(\Omega)$. This is what we wanted.

(4) Let Ω be a smooth bounded domain and $b \in \mathbb{R}^n$ be a fixed vector. Show that there exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C\|b \cdot \nabla u\|_{L^2(\Omega)}$$

for all $u \in H_0^1(\Omega)$. Explain why this doesn't hold for all $u \in H^1(\Omega)$.

Proof: We can pick coordinates such that b is a multiple of e_1, and without loss of generality, we can assume that $b = e_1$. Then $b \cdot \nabla u = u_{x_1}$. We can write $x = (x_1, x')$.

Now suppose $u \in C_c^\infty(\Omega)$. We can extend u by zero to all of $\mathbb{R}^n$. Since Ω is bounded, there exists c such that for each $x \in \Omega$, $x - ce_1 \notin \Omega$. Therefore for each $x \in \Omega$

$$u(x) = \int_{x_1-c}^{x_1} u_{x_1}(s, x')ds$$

Then

$$\begin{aligned}|u(x)|^2 &\leq C\int_{x_1-c}^{x_1}|u_{x_1}(s,x')|^2ds\\ &\leq C\int_{-\infty}^{\infty}|u_{x_1}(s,x')|^2ds\end{aligned}$$

so integrating both sides over Ω,

$$\|u(x)\|^2_{L^2(\Omega)} \leq C_\Omega \int \|u_{x_1}\|^2_{L^2(\Omega)}.$$

This holds for $u \in C_c^\infty(\Omega)$. But by the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ it extends to all $u \in H_0^1(\Omega)$, and so this is what we wanted to show.

It does not hold for all $u \in H^1(\Omega)$, since if u is a non-zero constant then the right side is zero but the left side is not.

(5) Let L be the differential operator defined by $Lu = b \cdot \nabla u + cu$, where b and c are constant. Suppose that for any domain Ω of the form

$$\Omega = \{x \in \mathbb{R}^n | x_n > f(x_1, \ldots, x_{n-1})\},$$

where f is smooth, there exists a constant $C_\Omega > 0$ independent of b such that the inequality

$$\|u\|_{L^2(\Omega)} \leq C_\Omega(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)})$$

holds for all $u \in C^1(\Omega) \cap H^1(\Omega)$. Show then that for any smooth bounded domain Ω, a similar inequality holds provided that b is sufficiently small.

Proof: Let Ω be a smooth bounded domain. For each $x_0 \in \partial\Omega$, there exists a neighbourhood U of x_0 for which we can pick coordinates such that $U \subset \partial\Omega$ is a subset of a graph $\{x_n = f(x_1, \ldots, x_{n-1})\}$ and $U \subset \Omega$ lies in the set

$$\{x \in \mathbb{R}^n | x_n > f(x_1, \ldots, x_{n-1})\}.$$

Since $\partial\Omega$ is compact, we can cover it with finitely many such neighbourhoods $\{U_1, \ldots U_m\}$. Adding another set U_0 to cover the interior of Ω if necessary, we can take a partition of unity $\chi_0, \ldots \chi_m$ subordinate to $\{U_0, U_1, \ldots U_m\}$, and write

$$u = \sum_{k=0}^m \chi_k u =: \sum_{k=0}^m u_k.$$

Then each u_k is supported in a domain of the form given in the question, so

$$\begin{aligned}\|u_k\|_{L^2(\Omega)} &\leq C_\Omega(\|Lu_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\partial\Omega)})\\ &\leq C_\Omega(\|Lu_k\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)})\end{aligned}$$

Now

$$Lu_k = (b \cdot \nabla \chi_k)u + \chi_k Lu,$$

so

$$\|u_k\|_{L^2(\Omega)} \leq C_\Omega(\|Lu\|_{L^2(\Omega)} - |b|\|\nabla\chi_k\|_{L^\infty}\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}).$$

Adding up over all k gives

$$\|u\|_{L^2(\Omega)} \leq (m+1)C_\Omega(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}) - (m+1)C_\Omega|b|\|\nabla\chi_k\|_{L^\infty}\|u\|_{L^2(\Omega)}$$

and for small enough b depending only on Ω we can hide the last term in the right side of the inequality.