# Problem Set 2

(1) Reading: Read Section 5.3-5.4 of Evans. Section 5.8.5 might also be helpful for problem 4.

(2) Do problems 5 and 6 from section 5.10 of Evans.

(3) Let $U$ and $V$ be bounded open sets in $\mathbb{R}^n$, and suppose that $\Phi : U \to V$ is a smooth bijection. Show that if $u \in W^{k,p}(V)$ for some $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, then $u \circ \Phi \in W^{k,p}(U)$.

(4) (An alternative definition of $H^k$). Recall that for $k \in \{0\} \cup \mathbb{N}$,

$$H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$$

by definition. Show that $u \in L^2(\mathbb{R}^n)$ is in $H^k(\mathbb{R}^n)$ if and only if

$$(1 + |\xi|^k)\hat{u}(\xi) \in L^2(\mathbb{R}^n)$$

where $\hat{u}$ is the Fourier transform of $u$, and moreover that there exists some constant $C$ such that

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |\xi|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}.$$

Note that we can use this to extend the definition of $H^k$ to arbitrary $k \in \mathbb{R}$, by defining

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) | (1 + |\xi|^k)\hat{u} \in L^2(\mathbb{R}^n)\}$$

with

$$\|u\|_{H^k(\mathbb{R}^n)} = \|(1 + |\xi|^k)\hat{u}\|_{L^2(\mathbb{R}^n)}$$

for all $k \in \mathbb{R}$.

(5) Suppose that $u \in C_c^\infty(\mathbb{R}^n)$, and let $P = \{(x_1, \dots x_n) \in \mathbb{R}^n | x_n = 0\}$. Show that

$$\|u\|_{H^{\frac{1}{2}}(P)} \leq C\|u\|_{H^1(\mathbb{R}^n)}$$

for some constant $C$. Conclude that the map $T : u \mapsto u|_P$ extends to a bounded function

$$T : H^1(\mathbb{R}^n) \to H^{\frac{1}{2}}(P).$$

In particular, this implies that for every function in $H^1(\mathbb{R}^n)$, the restriction to a hyperplane is well defined. Note that this is not true for $L^2(\mathbb{R}^n)$!