Problem Set 4

(1) Reading: Read Section 5.6 and Appendix D.1-D.3 of Evans.

(2) Let $p > n$ and prove that if $u \in W^{1,p}(\mathbb{R}^n)$, then there exists a continuous function u^* such that $u^* = u$ almost everywhere. (We discussed this in class but didn't provide a detailed proof.)

(3) Use the Fourier transform characterization of $H^k(\mathbb{R}^n)$ to prove that for k sufficiently large, any $u \in H^k(\mathbb{R}^n)$ is continuous. (Hint: recall that the Fourier transform of an L^1 function is continuous.)

(4) Theorem 5.6.3 says in particular that if $U \subset \mathbb{R}^3$ is bounded and open, then there exists C depending only on U such that

$$\|u\|^2_{L^2(U)} \leq C\|\nabla u\|^2_{L^2(U)}$$

for all $u \in H^1_0(U)$. Let $C(U)$ be the infimum of all C for which this statement holds. Now suppose u solves the reaction-diffusion equation

$$\partial_t u = \triangle u + qu \text{ on } U$$

where $q > 0$ is a fixed number, with the boundary condition $u|_{\partial U \times \{t>0\}} = 0$. Show that

$$\int_U u^2(t, x)dx$$

is a decreasing function of t if $q < 1/C(U)$.

The proofs of the trace theorem, the Gagliardo-Nirenberg-Sobolev inequality, and Morrey's inequality are quite complicated. To understand them, it helps to try to write an outline of the proof which describes the main ideas, without the details. Here is an example of the sort of thing I mean, applied to the extension theorem (5.3.3):

Theorem 0.1. *Assume that U is bounded and ∂U is C^1. Select a bounded open set V such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$, $Eu = u$ inside U, and Eu has support within V.

Proof idea: First, look at a neighbourhood of a point x^0 on the boundary, and suppose ∂U is flat near x^0. Then we can pick coordinates so that near

x^0, ∂U lies in the plane $x_n = 0$ and U lies in $x_n > 0$. (If it's not flat, then we can use a change of variables to ensure that it becomes flat).

Second, given a function $u \in C^1(U)$, we can write down a sort of reflection of u across the plane $x_n = 0$: we write

$$\bar{u}(x) = \begin{matrix} u(x) & x_n \geq 0 \\ au(x_1, \ldots, -x_n) + bu(x_1, \ldots, -cx_n) & x_n < 0 \end{matrix}$$

where a, b, and c are chosen so that u and $\partial_n u$ are continuous across the plane $x_n = 0$. This allows us to extend u past ∂U, at least locally, and then we can multiply by a cutoff function to ensure that it's smooth and supported in V.

Third, we can use a partition of unity to cut up a general function u into pieces which are each supported in small neighbourhoods of the boundary. Then we can apply the above construction to each piece and then add the pieces back together.

This summary describes just a few main ideas, and leaves out all the details. It's not the only possible summary, and it's certainly not a real proof! But if you had just a few minutes to explain the proof, you might end up saying something like this, and you could probably reconstruct the real proof from this summary with a little work.

(5) Write down summaries of Theorem 5.5.1, 5.6.1, and 5.6.4 in the spirit of the above summary – imagine you have only a few minutes to describe the gist of proof, and see if you can explain the main ideas.