Problem Set 5

(1) Reading: Read Section 6.1 and 6.2 to the end of 6.2.2 (page 320).

(2) Do problems 2 and 3 in Section 6.6 of Evans.

There is a characterization of H^{-1}, defined as the dual space to H_0^1, given in section 5.9 (pages 299-300). You should read this, but I prefer to think of H^{-1} in a different way.

(3) Suppose Ω is open and has C^1 boundary. Define

$$H^{-1}(\Omega) = \{u \in L^1_{loc}(\Omega) | \int_\Omega uv < \infty \text{ for all } v \in H_0^1(\Omega)\}$$

with the dual norm

$$\|u\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|\int_\Omega uv|}{\|v\|_{H^1(\Omega)}}.$$

Explain why $H^{-1}(\Omega)$ is isomorphic to the dual space of $H_0^1(\Omega)$, as Banach spaces.

(4) Show that for $u \in L^2(\Omega)$, we have $u \in H^{-1}(\Omega)$ and

$$\|u\|_{H^{-1}(\Omega)} \leq \|u\|_{L^2(\Omega)}.$$

Moreover show that $|x|^{-\frac{1}{2}}$ is in $H^{-1}(-1,1)$ but not $L^2(-1,1)$, so that L^2 is in general a strict subset of H^{-1}.

(5) Show that if $u \in C^1(\overline{\Omega}) \cap L^2(\Omega)$, then

$$\|\partial_{x_1} u\|_{H^{-1}(\Omega)} \leq \|u\|_{L^2(\Omega)}.$$

Conclude that ∂_{x_1} extends to a map $\partial_{x_1} : L^2(\Omega) \to H^{-1}(\Omega)$. The moral of this story is that H^{-1} contains the "derivatives of L^2 functions".

(6) Suppose Ω is bounded and open, with C^1 boundary, and show that if $f \in H^{-1}(\Omega)$, then there exists a weak solution $u \in H_0^1(\Omega)$ to the problem

$$\begin{aligned} \triangle u &= f \text{ on } \Omega \\ u|_{\partial\Omega} &= 0. \end{aligned}$$