Problem Set 6

(1) Reading: Read Section 5.7 of Evans, and Section 5.8 up to the end of the proof of Theorem 1 (page 290). Also read Section 6.2.3 of Evans (page 320-325).

(2) Do question 4 from Section 6.6 of Evans (you will probably need Theorem 1 in Section 5.8 to do this)

(3) Let U be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, and consider the problem

$$\begin{aligned} -\triangle u &= f \text{ on } U \\ u|_{\partial U} &= 0 \end{aligned}$$

As discussed in class, this has a weak solution $u \in H_0^1(\mathbb{R}^n)$ for each $f \in L^2(U)$. Define the map $L^{-1} : L^2(U) \to H_0^1(U)$ by $L^{-1}f = u$. Show that the map L^{-1}, viewed as a map from $L^2(U)$ to $L^2(U)$, is compact.

(4) Let U be a bounded open set in $\mathbb{R}^n$, $n \geq 2$. Let Σ be the set of $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} -\triangle u - \lambda u &= 0 \text{ on } U \\ u|_{\partial U} &= 0 \end{aligned}$$

has a nonzero weak solution $u \in H_0^1$. Using the previous question show that Σ is countable and can be arranged in an increasing sequence with no finite limit point. Explain why

$$\begin{aligned} -\triangle u - \lambda u &= f \text{ on } U \\ u|_{\partial U} &= 0 \end{aligned}$$

has a unique weak solution for each $f \in L^2(U)$ whenever $\lambda \notin \Sigma$.

(5) Let $U = [0, \pi] \times [0, \pi] \subset \mathbb{R}^2$, and define Σ as in the previous question. Show that

$$\Sigma = \{(n^2 + m^2) | n \in \mathbb{N}\}.$$

(Hint: Use Fourier series on the square. You may need to consult problem set 10 from last semester).

In addition to the Riesz representation theorem, the Lax-Miller theorem, the Fredholm alternative, and the associated facts about compact operators, there are at least two other important functional analysis results which imply important existence results for PDE. One of them is the Hahn-Banach theorem:

Theorem 0.1. *Let X be a (real) Banach space, and suppose $Y \subset X$ is a vector subspace of X. Suppose $\varphi : Y \to \mathbb{R}$ is a bounded linear functional, with the bound*

$$|\varphi(y)| \leq c\|y\|$$

for all $y \in Y$. Then there exists a bounded linear functional $f^* : X \to \mathbb{R}$ such that $\varphi^*(y) = \varphi(y)$ for all $y \in Y$, and

$$|\varphi^*(x)| \leq c\|x\|$$

for all $x \in X$.

The Hahn-Banach theorem is easy to prove if $X = \mathbb{R}^n$, and not too much harder when X is a Hilbert space with countable basis. But in general the proof of the Hahn-Banach theorem requires the axiom of choice. (!)

(6) Let b be a fixed unit vector in $\mathbb{R}^n$, and set $\mathcal{L}_\pm = \triangle \pm b \cdot \nabla + 1$. It can be shown that if U is a bounded domain, then

$$\|\mathcal{L}_\pm u\|_{L^2(U)} \geq C\|u\|_{L^2(U)}$$

for all $u \in C_c^\infty(U)$. Assuming this inequality for now, consider the subspace $Y \subset L^2(U)$ defined by

$$Y = \{v \in L^2(U) | v = \mathcal{L}_+ w \text{ for some } w \in C_c^\infty(U)\}.$$

Let $f \in L^2$. Let $\varphi : Y \to \mathbb{R}$ be defined by

$$\varphi(v) = (w, f)$$

where $\mathcal{L}_+ w = v$. Show that φ is well defined (does not depend on a choice of w) and that φ is a bounded linear functional on Y.

(7) Define $u \in L^2(U)$ to be a super-weak solution to

$$(C) \qquad \begin{cases} \mathcal{L}_- u & = f \text{ on } U \\ u|_{\partial U} & = 0 \end{cases}$$

if

$$(u, \mathcal{L}_+ v) = (f, v)$$

for all $v \in C_c^\infty(U)$ (why does this make sense?) Use the previous question, together with Hahn-Banach, to show that (C) has a super-weak solution $u \in L^2(U)$ for each $f \in L^2(U)$, with the bound

$$\|u\|_{L^2(U)} \lesssim \|f\|_{L^2(U)}.$$