Problem Set 7

(1) Reading: Read Section 5.8.2 of Evans (pages 291-293), and Section 6.3 up to the end of the proof of Theorem 3 (page 334).

(2) Do question 12 from Section 5.10 of Evans and question 5 and 7 from Section 6.6 of Evans.

(3) Let U be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary. Consider the Neumann problem

$$\begin{aligned} -\triangle u &= f - u \text{ on } U \\ \partial_\nu u|_{\partial U} &= 0 \end{aligned}$$

where ν is the unit outward normal vector. Using the definition of weak solution from question 4 of Section 6.6 of Evans, show that this problem has a unique weak solution $u \in H^1(U)$ for each $f \in L^2(U)$.

(4) Let U be as in the previous question. Show that there exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the problem

$$\begin{aligned} -\triangle u &= \lambda u + f \text{ on } U \\ \partial_\nu u|_{\partial U} &= 0 \end{aligned}$$

has a unique weak solution $u \in H^1(U)$ if and only if $\lambda \notin \Sigma$; show also that if Σ is not finite then it can be arranged in a non-decreasing sequence which goes to infinity. Note that $0 \in \Sigma$, by question 4 of Section 6.6 of Evans.